**THE AUSTRALIAN NATIONAL UNIVERSITY**
**DEPARTMENT OF STATISTICS AND ECONOMETRICS**

# STATISTICAL INFERENCE - STAT3013/STAT8027

**Mid-Semester Examination 2000**

**Total Marks: 50**

*Reading Period: 15 Minutes*
*Time Allowed: Two Hours*
*Permitted Materials: Course Brick, Lecture Notes, Non-Programmable Calculator*

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**Question 1**

Indicate whether each statement is either TRUE or FALSE (please write the entire word, as hasty F's can sometimes look like T's). Two marks will be awarded for each correct response, two marks will be deducted for each incorrect response, and zero marks will be given for no response.

**(a)** Generally, it is not possible to find an estimator of a parameter $\tau(\theta)$ which has uniformly smallest *MSE* for all possible parameter values $\theta \in \Theta$.

**(b)** If an unbiased estimator, $T$, of a parameter $\tau(\theta)$ has a variance which exceeds the Cramér-Rao bound, then $T$ cannot be the *UMVU* estimator of $\tau(\theta)$.

**(c)** Suppose that $T = t(X_1, \ldots, X_n)$ is an estimator of $\tau(\theta)$. Employing squared-error loss and a prior distribution $\pi(\theta)$, the Bayes risk of $T$, $r_{\pi,\ell}(t)$, is uniformly smaller than the *MSE* of $T$; that is, $r_{\pi,\ell}(t) \leq MSE_t(\theta)$ for all $\theta \in \Theta$.

**(d)** Let $T_1 \neq T_2$ be two distinct, unbiased estimators of a parameter $\tau(\theta)$. If $S$ is a complete and sufficient statistic for $\theta$, then $E(T_1|S) = E(T_2|S)$.

**(e)** The Bayes estimator under absolute-error loss for a parameter $\tau(\theta)$ is the median of the posterior distribution of $\theta$, $\pi(\theta|X_1, \ldots, X_n)$.

**[10 marks]**

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**Question 2**

Let $X_1, \ldots, X_n$ be an *iid* sample from a Normal distribution with mean $\theta$ and variance $\theta$ (i.e., the mean and variance are equal).

**(a)** Determine a minimal sufficient statistic. **[5 marks]**

**(b)** Determine the Cramér-Rao bound for the variance of unbiased estimators of $\theta$. Does the variance of $\overline{X} = n^{-1} \sum_{i=1}^{n} X_i$ achieve the Cramér-Rao bound for any $\theta > 0$? **[6 marks]**

**(c)** Is $\overline{X}$ the *UMVU* estimator? Why or why not? If not, discuss how you would find the *UMVU* estimator. **[5 marks]**

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**Question 3**

We are considering a population of individuals each with a particular measure of "risk aversion" (i.e., a quantity between 0 and 1 which indicates the degree to which an individual seeks out risky behaviour). Suppose that we believe that the risk aversion measures for a randomly chosen person in the population follows a distribution with density function $f_X(x;\theta) = \theta x^{\theta-1}$ for $0 \le x \le 1$ and $\theta > 0$.

**(a)** Suppose that we observe a random sample of size $n$ from this population, $X_1, \ldots, X_n$. If we employ a Gamma prior distribution with shape parameter $\alpha$ and scale parameter $\alpha^{-1}$, so that

$$\pi(\theta) = \frac{\alpha^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\alpha\theta},$$

show that the posterior distribution of $\theta$ is Gamma with shape parameter $\alpha + n$ and scale parameter $\{\alpha - \sum_{i=1}^n \ln(X_i)\}^{-1}$. **[4 marks]**

**(b)** Find the Bayes estimator of $\theta$ with respect to squared-error loss. **[4 marks]**

**(c)** It can be shown that the *MLE* of $\theta$ based on the sample $X_1, \ldots, X_n$ is given by $\hat{\theta}_{MLE} = -n\{\sum_{i=1}^n \ln(X_i)\}^{-1}$. Use this fact to discuss the form of the posterior Bayes estimator in the situation that $n$ tends to infinity for a fixed $\alpha$. Further, discuss the form of the posterior Bayes estimator in the situation that $\alpha$ tends towards infinity for a fixed sample size $n$. **[4 marks]**

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**Question 4**

Suppose that $X_1, X_2, X_3$ is an *iid* sample from a bivariate distribution $F$; that is, the $X_i = (X_{1i}, X_{2i})$ values are ordered pairs:

$$X_1 = (X_{11}, X_{21}) = (1, 4), \quad X_2 = (X_{12}, X_{22}) = (3, 2), \quad X_3 = (X_{13}, X_{23}) = (4, 6).$$

Define $\mu(F) = E_F(X_i) = \{E_F(X_{1i}), E(X_{2i})\} = \{\mu_1(F), \mu_2(F)\}$. Also, let $\overline{X}_1 = \frac{1}{3}\sum_{i=1}^3 X_{1i}$ and $\overline{X}_2 = \frac{1}{3}\sum_{i=1}^3 X_{2i}$. We are interested in estimating the quantity $\theta = \theta(F) = \mu_1(F)/\mu_2(F)$ and we are considering the two estimators $T_1 = \theta(\hat{F}) = \overline{X}_1/\overline{X}_2$ and $T_2 = \frac{1}{3}\sum_{i=1}^3 (X_{1i}/X_{2i})$ [i.e., $T_1$ is the ratio of the sample averages and $T_2$ is the average of the sample ratios].

**(a)** Find the Jackknife estimates of bias for $T_1$ and $T_2$. **[5 marks]**

**(b)** Find the Jackknife estimates of variance for $T_1$ and $T_2$. **[5 marks]**

**(c)** Which estimator would you prefer? Why? **[2 marks]**

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*END OF EXAMINATION*