Exercise H5.1 The following exercise presupposes the the definition of a sufficient statistic $T = T(X)$ in a continuous Model C, cf. section 4.3: the observed random variable $X = (X_1, \dots, X_k)$ is continuous with values in $\mathbb{R}^k$ and $\mathcal{L}(X) \in \{P_\vartheta, \vartheta \in \Theta\}$. Each law P_ϑ is described by a joint density $p_\vartheta(x) = p_\vartheta(x_1, \dots, x_k)$, and $\Theta \subseteq \mathbb{R}^d$.

Suppose T is real valued and there are random variables $Y_2, \dots, Y_k$, all functions of X, such that the r.v.'s $(T, Y_2, \dots, Y_k)$ have a joint density $q_\vartheta(t, y_2, \dots, y_k)$ in $\mathbb{R}^k$. Then T is **sufficient** for $\vartheta \in \Theta$ if there is a version of

$$q_\vartheta(y_2, \dots, y_k|t)$$

which does not depend on ϑ.

Consider the **Gaussian scale model:** $X_1, \dots, X_n$ are i.i.d. random variables, each having density of $N(0, \sigma^2)$, with unknown $\vartheta = \sigma^2 > 0$. In case $n = 2$, find a sufficient statistic (real valued) for σ^2.

Hint: consider a polar coordinate transformation of X_1, X_2, cf. Durrett[1], Chap. 3.2, Example 2.4, p. 102.

Solution: for X_1 and X_2 i.i.d. random variables with densities $N\left(0, \sigma^2\right)$, we have the joint density

$$p_\theta\left(x_1, x_2\right) = \frac{1}{2\pi\theta} \exp\left(-\frac{x_1^2 + x_2^2}{2\theta}\right),$$

where $\theta = \sigma^2$.

By using the polar coordinate transformation, we have

$$p_\theta^p\left(r, \omega\right) = \frac{1}{2\pi\theta} \exp\left(-\frac{r^2}{2\theta}\right) r,$$

where the transformation is $x_1 = r\cos\left(\omega\right)$, and $x_2 = r\sin\left(\omega\right)$.

Note that

$$\begin{aligned} p_\theta^p\left(r \int_0^{2\pi}, \omega\right) d\omega &= \int_0^{2\pi} \frac{1}{2\pi\theta} \exp\left(-\frac{r^2}{2\theta}\right) r d\omega \\ &= \frac{r}{\theta} \exp\left(-\frac{r^2}{2\theta}\right), \end{aligned}$$

and,

[1] Richard Durrett, The Essentials of Probability, Duxbury Press, 1994.

$$p_\theta^p(\omega|r) = \frac{p_\theta^p(r,\omega)}{p_\theta^p\left(r\int_0^{2\pi},\omega\right)d\omega}$$
$$= \frac{1}{2\pi},$$

which is independent of θ.

This implies $\left(x_1^2+x_2^2\right)^{\frac{1}{2}}$ is a sufficient statistics.

Note: I apologize for giving some students hard time about what we need to prove in this problem. Professor Nussbaum told me we don't need to prove the equivalence between this definition of sufficiency and the generalization of the definition given in our notes. He also told me we need a little modification about the definition.

(Comment: the condition which should have been made explicit in the problem is that the $k-$dimensional r.v. $(T, Y_2, \ldots, Y_k)$ is a one to one function of the $k-$dimensional r.v. X. (M. N.)).

Exercise H5.2 Consider a bivariate normal random variable $Z = (X, Y)$ with law $N_2(\mathbf{0}, \Sigma)$, where

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}$$

and $\sigma_X^2\sigma_Y^2 - \sigma_{XY}^2 > 0$ (this condition ensures that Σ is a covariance matrix, see next exercise below). Find a function h_0 of X such that

$$E\left(Y - h_0(X)\right)^2 = \min_h E\left(Y - h(X)\right)^2$$

where the minimum is taken over all functions h of X (then $h_0(X)$ is then called a **best predictor** of Y).

Hint: generalize from Bayes estimation in the Gaussian location model with $n = 1$ (the case where $Y = \vartheta$, $X = \vartheta + \xi$, and ϑ, ξ are independent normal $N(0, \tau^2)$, $N(0, \sigma^2)$).

Solution: following the notes in page 9 (or the idea in the proof of Rao-Blackwell), we have

$$E\left(Y - h(X)\right)^2 \geqslant E\left(Y - E(Y|X)\right)^2,$$

and $h_0(x) = E(Y|X)$.

From the covariance matrix, we know the joint density of X and Y is

$$p(x,y) = \frac{1}{2\pi|\sum|^{\frac{1}{2}}}\exp\left(\sigma_X^2x^2 - 2\sigma_{XY}xy + \sigma_Y^2y^2\right).$$

It is easy to see that the distribution of $Y|X$ is a normal distribution with mean $\frac{\sigma_{XY}}{\sigma_X^2}x$ (trivial, similar to homework 3).

This implies $h_0(x) = E(Y|X) = \frac{\sigma_{XY}}{\sigma_X^2}x$.

Exercise H5.3. Let Σ be a matrix as above, i.e.

$$\Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}$$

where $\sigma_X^2 > 0$, $\sigma_Y^2 > 0$, and $|\Sigma| \neq 0$.. Show that Σ is the covariance matrix of a random vector in $\mathbb{R}^2$ if and only if

$$\sigma_X^2 \sigma_Y^2 - \sigma_{XY}^2 > 0 \tag{1}$$

Hint: In view of the facts in section 5 handout, it suffices to show that the condition (1) is equivalent to Σ being positive definite. If you know a more general fact from matrix theory which implies the claim, its statement suffices.

Solution: from the basic linear algebra, we have the symmetric matrix is positive definite if and only if

λ_1 and λ_2 are positive, where λ_1 and λ_2 are eigenvalues of Σ.

Note that $\lambda_1 > 0$, $\lambda_2 > 0 \Leftrightarrow \lambda_1 + \lambda_2 > 0$, $\lambda_1 \lambda_2 > 0$

$$\Leftrightarrow \lambda_1 \lambda_2 > 0 \quad (\text{because } \lambda_1 + \lambda_2 = tr\,(\Sigma) = \sigma_X^2 + \sigma_Y^2 > 0)$$

$$\Leftrightarrow |\Sigma| > 0 \quad (\text{because } |\Sigma| = \lambda_1 \lambda_2)\,.$$

This implies Σ is positive if and only if $\sigma_X^2 \sigma_Y^2 - \sigma_{XY}^2 > 0$.