Exercise H6.1 Let $X_1, \dots, X_{n_1}$, be independent $N(\mu_1, \sigma^2)$ and $Y_1, \dots, Y_{n_2}$ be independent $N(\mu_2, \sigma^2)$, also independent of $X_1, \dots, X_{n_1}$ (n_1,$n_2 > 1$) For each of the two samples, form the sample means $\bar{X}$, $\bar{Y}$ and the bias corrected sample variances

$$S^2_{(1)} = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (X_i - \bar{X})^2,\ S^2_{(2)} = \frac{1}{n_2 - 1} \sum_{i=1}^{n_2} (Y_i - \bar{Y})^2$$

Consider the statistic

$$Z = \frac{\bar{X} - \bar{Y}}{\sigma \left(\frac{1}{n_1} + \frac{1}{n_2} \right)^{1/2}}$$

which is standard normal if $\mu_1 = \mu_2$. In a model where μ_1, μ_2 are unknown but σ^2 is known, it obviously can be used to build a confidence interval for the difference $\mu_1 - \mu_2$.

For the case that in addition σ^2 is unknown, find a statistic which has a t-distribution if $\mu_1 = \mu_2$ (this would then be called a "studentized" statistic) , and find the degrees of freedom.

Solution: since $\overline{X} - \overline{Y}$ is a random variable according to a normal distribution with $E\left(\overline{X} - \overline{Y}\right) = \mu_1 - \mu_2$ and $Var\left(\overline{X} - \overline{Y}\right) = Var\left(\overline{X} - \overline{Y}\right) + Var\left(\overline{X} - \overline{Y}\right) = \sigma^2 \left(\frac{1}{n_1} + \frac{1}{n_2}\right)$, then

$$\mathcal{L}\left(\frac{\overline{X} - \overline{Y}}{\sigma\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{\frac{1}{2}}}\right) = N\left(\mu_1 - \mu_2, 1\right).$$

And from Theorem 6.2, we know

$$\mathcal{L}\left(\frac{n_1 - 1}{\sigma^2}\widehat{S}^2_{(1)}\right) = \chi^2\left(n_1 - 1\right),$$

and

$$\mathcal{L}\left(\frac{n_2 - 1}{\sigma^2}\widehat{S}^2_{(2)}\right) = \chi^2\left(n_2 - 1\right),$$

then

$$\mathcal{L}\left(\frac{n_1 - 1}{\sigma^2}\widehat{S}^2_{(1)} + \frac{n_2 - 1}{\sigma^2}\widehat{S}^2_{(2)}\right) = \chi^2\left(n_1 + n_2 - 2\right).$$

This implies

$$\mathcal{L}\left(\frac{\dfrac{\overline{X}-\overline{Y}}{\sigma\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)^{\frac{1}{2}}}(n_1+n_2-2)^{\frac{1}{2}}}{(\dfrac{n_1-1}{\sigma^2}\widehat{S}^2_{(1)}+\dfrac{n_2-1}{\sigma^2}\widehat{S}^2_{(2)})^{\frac{1}{2}}}\right)=t\left(n_1+n_2-2\right)$$

i.e.,

$$\mathcal{L}\left(\frac{(\overline{X}-\overline{Y})\left(n_1+n_2-2\right)^{\frac{1}{2}}}{\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)^{\frac{1}{2}}\left((n_1-1)\widehat{S}^2_{(1)}+(n_2-1)\widehat{S}^2_{(2)}\right)^{\frac{1}{2}}}\right)=t\left(n_1+n_2-2\right).$$

Exercise H6.2 Show that the t-distribution with n degrees of freedom has

a) finite moments of up to order $n-1$

b) no moments of order n and higher.

Solution: in page 58 of our notes, we know the density of the law t_n is

$$f_n\left(x\right)=c\left(1+\frac{x^2}{n}\right)^{-(n+1)/2},$$

where c is a positve constant.

We need to prove

$$\int_{-\infty}^{+\infty} f_n\left(x\right)x^k dx=c\int_{-\infty}^{+\infty}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}x^k dx$$

exists for $k\leqslant n-1$, and doesn't exist for $k\geqslant n$.

Note that this function is continous on R, so it enough to show

$$\int_{1}^{+\infty}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}x^k dx$$

exists for $k\leqslant n-1$, and doesn't exist for $k\geqslant n$.

Since

$$\lim_{x\to+\infty}\frac{\left(1+\dfrac{x^2}{n}\right)^{-(n+1)/2}x^k}{x^{k-(n+1)}}=n^{(n+1)/2},$$

then it is enough to see

$$\int_1^{+\infty} x^{k-(n+1)}dx$$

exists or not.

Obviously, $\int_1^{+\infty} x^{k-(n+1)}dx$ exists for $k \leqslant n-1$, and doesn't exist for $k \geqslant n$.

This implies

$$\int_1^{+\infty} \left(1+\frac{x^2}{n}\right)^{-(n+1)/2} x^k dx$$

exists for $k \leqslant n-1$, and doesn't exist for $k \geqslant n$.

Exercise H6.3 For the following problem, assume that $n \geq 2$ and consider the unit sphere in $\mathbb{R}^n$

$$K_n = \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Assume that there is a one-to-one-mapping h from a set $B \subset \mathbb{R}^{n-1}$.onto K_n such that every vector x in $\mathbb{R}^n$ with $\|x\| > 0$ can be written

$$x = \|x\| \cdot h(u_x)$$

where $u_x \in B$. Assume also that the mapping $(r, u) \mapsto r \cdot h(u)$, $r > 0$, $u \in B$ allows a change of variables with Jacobian $C_n r^{n-1}$: if $f(x)$ is any density in $x = (x_1, \dots, x_n)$ then the density in new variables $(r, u) = (\|x\|, u_x)$ is

$$C_n \cdot f(r \cdot h(u)) \cdot r^{n-1}$$

where C_n is a constant depending only on n.

Consider now a Gaussian scale model: $X_1, \dots, X_n$ are independent and identically distributed random variables each having law $N(0, \sigma^2)$, where $\sigma^2 > 0$ is unknown. Set $X = (X_1, \dots, X_n)^\top$.

(i) Note that the uniform law Q on the sphere K_n is the unique probability measure on K_n which is invariant under orthogonal transformations M of $\mathbb{R}^n$:

$$Q(A) = Q(MA) \text{ for every measurable set } A \subset K_n.$$

Show that the random vector $X/\|X\|$ has the distribution Q.

(ii) Show that the random variables $\|X\|^2$ and $U = u_X = h^{-1}(X/\|X\|)$ are independent.

(iii) Show that $\|X\|^2$ is a sufficient statistic for σ^2.

Remark: (iii) is a generalization of Exercise H5.1 to $n \geq 2$; there also sufficient statistics for continuous models were defined. This previous definition should be completed by the assumption that the random vector $(T, Y_2, \dots, Y_k)$ introduced there is a one-to-one function of the random vector X.

Solution: **(i)** Since

$$P(X/\|X\| \in A) = P(MX/\|X\| \in MA)$$
$$= P(MX/\|MX\| \in MA) \text{ (because } M \text{ is orthogonal)}$$
$$= P(X/\|X\| \in MA) \text{ (because } \mathcal{L}(X) = \mathcal{L}(MX) \text{ from Lemma 5.8)},$$

then $X/\|X\|$ has the distribution Q.

(ii) since the the joint density of X is

$$f(x_1, ..., x_n) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} \textstyle\sum_{i=1}^n x_i^2\right),$$

then the density in the new variable (r, u) is

$$g(r, u) = C_n \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\frac{1}{2\sigma^2} r^2\right) r^{n-1},$$

which is independent of u.

Thus the marginal density of u is a constant. Thus $g(r, u)$ is a product of the marginal densities of r and u.

This implies $\|X\|^2 = r^2$ and u are independent.

(iii) note that the polar coordinates gives the mapping $(r, u) \mapsto r \cdot h(u)$, $r > 0$ which allows a change of variables with Jacobian $C_n r^{n-1}$. And from (i) and (ii), we know $p(u|r)$ is independent of σ^2. This implies $||X||^2$ is a sufficient statistics for σ^2.