

Separation of Variables and the First Controlled Nuclear Reaction

William F. Moss*

1 Introduction

In this lesson we will discuss the events leading up to the creation by Enrico Fermi and others of the first controlled nuclear reaction in 1942. We will look at how Fermi used separation of variables to solve a diffusion model of his reactor and how he estimated the reactor size that would allow for a self-sustaining reaction. We will also mention some related events and dates during WWII.

2 Timeline

1938 Fermi leaves Italy for Sweden to receive the Nobel Prize for physics and never returns to Italy.

January 1939 Fermi arrives at Columbia University in New York and receives news that Hahn and Strassman in Berlin have split the uranium atom by bombarding it with neutrons. This process becomes known as fission. Fermi immediately recognizes the possibilities for a bomb and for power generation.

January – July 1939 Fermi and others study the possibility for creating a self-sustaining chain reaction. The cycle ultimately is shown to have the following steps.

- Thermal neutrons are absorbed by uranium causing fission.
- High energy fission products and fast neutrons are released.
- Fission products and neutrons slow down by collisions with the surrounding material. Carbon, and later heavy water, are found to be good moderators; that is, materials that can slow fast neutrons to thermal energies without absorbing them.

*Department of Mathematical Sciences, Clemson University, Clemson, SC 29634–0975, U.S.A. (bmoss@clemson.edu). Copyright ©2001 William F. Moss. All rights reserved.

Summer 1939 Fermi's colleagues Szilard and Wigner visit Einstein. Einstein writes his famous letter to President Roosevelt.

Summer 1940 Szilard recommends that all uranium research be kept secret fearing that Germany would get an atomic bomb first.

September 1940 Carbon is found to be a good moderator.

November 1941 A lattice of uranium oxide lumps embedded in graphite is studied. It is found that if the dimensions of the lattice are sufficiently large, a divergent chain reaction will occur.

December 6, 1941 Two reports of the National Academy of Sciences had previously indicated that a chain reaction could be used to produce power or to produce plutonium, a likely competitor to uranium as a material for making an atomic bomb. The National Defense Research Committee announced an all-out effort.

December 7, 1941 Pearl Harbor was attacked and the United States entered the war against Germany, Italy, and Japan.

November 1942 A British commando raid fails to destroy Germany's heavy water production facility in occupied Norway.

December 2, 1942 In a hand-ball court under the West Stands at the University of Chicago, Fermi creates the first controlled, self-sustaining nuclear reaction using a carbon-uranium pile and cadmium control rods. The cadmium rods absorb neutrons. Once the last rod is removed from the reactor, a self-sustaining reaction takes place. A coded message is sent to the government, "The Italian navigator has just landed."

February 1943 A Norwegian scout commando raid fails to destroy the heavy water production facility.

November 1943 A U.S. Air Force bombing raid fails to destroy the heavy water production facility.

February 1944 Norwegian scouts sink the ferry taking the heavy water back to Germany.

3 Applying Separation of Variables to Fermi's Model

Fermi's research showed that for a self-sustaining reaction to take place the number of neutrons leaking out of the reactor needed to be minimized and that this could be accomplished with a spherical reactor. Fermi constructed his reactor in an approximate spherical shape using 8 inch graphite cubes and a wooden support structure. Uranium was placed in a hole drilled in each cube.

To simplify this presentation, we will find the solution to Fermi's mathematical model in Cartesian coordinates. Our reactor will be a box with side lengths a, b, c in the x, y, z directions. We will encounter a familiar regular Sturm-Liouville problem. If we were to solve this problem using spherical coordinates for a spherical reactor, we would encounter a singular Sturm-Liouville problem and an eigenfunction formula involving spherical Bessel functions. Both of these topics are outside the scope of this course.

The dependent variable ϕ in Fermi's mathematical model is called neutron flux and is proportional to the number of neutrons per unit volume in the reactor. Fermi's model looks like a heat conduction model except for a source term in the PDE of the form rate proportional to amount.

Problem: Find $\phi = \phi(x, y, z, t)$ so that

$$\text{PDE} \quad \frac{\partial \phi}{\partial t} - D \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right] = S\phi$$

$$\text{BC} \quad \phi = 0 \text{ on the boundary}$$

$$\text{IC} \quad \phi(x, y, z, 0) = f(x, y, z), \quad 0 < x < a, \quad 0 < y < b, \quad 0 < z < c.$$

S is a physical constant that is related to the relative amount of uranium and neutron absorbers in the reactor. D is a physical constant that is related to the scattering cross-section of the moderator.

We can now apply the method of separation of variables discussed in class over the last few weeks to Fermi's model just as he did.

3.1 Step 1: separate the homogeneous equations

Find nontrivial solutions to the homogeneous equations (PDE + BC) of the form

$$\phi(x, y, z, t) = X(x)Y(y)Z(z)T(t). \quad (1)$$

Substitute (1) into the PDE, we have

$$\begin{aligned} & X(x)Y(y)Z(z)T'(t) \\ & - D \left[X''(x)Y(y)Z(z)T(t) + X(x)Y''(y)Z(z)T(t) + X(x)Y(y)Z''(z)T(t) \right] \\ & = SX(x)Y(y)Z(z)T(t). \end{aligned}$$

Dividing by $X(x)Y(y)Z(z)T(t)$, we have

$$\frac{T'(t)}{T(t)} - D \left[\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} \right] = S. \quad (2)$$

Each term in (2) must be a constant since the variables x, y, z, t are independent. Set

$$\frac{X''(x)}{X(x)} = -\alpha, \quad \frac{Y''(y)}{Y(y)} = -\beta, \quad \frac{Z''(z)}{Z(z)} = -\delta \quad \text{and} \quad \lambda = \alpha + \beta + \delta.$$

Then

$$\frac{T'(t)}{T(t)} = S - D\lambda.$$

Next, we separate the BC. The boundary consists of the six sides of the box. We can break the BC into the following six equations and separate them.

$$0 = \phi(0, y, z, t) = X(0)Y(y)Z(z)T(t) \quad (3)$$

$$0 = \phi(a, y, z, t) = X(a)Y(y)Z(z)T(t) \quad (4)$$

$$0 = \phi(x, 0, z, t) = X(x)Y(0)Z(z)T(t) \quad (5)$$

$$0 = \phi(x, b, z, t) = X(x)Y(b)Z(z)T(t) \quad (6)$$

$$0 = \phi(x, y, 0, t) = X(x)Y(y)Z(0)T(t) \quad (7)$$

$$0 = \phi(x, y, c, t) = X(x)Y(y)Z(c)T(t) \quad (8)$$

To avoid a trivial solution we set the constant terms in (3) – (8) to zero.

Accumulating all the equations we have found by separation, we find three Sturm-Liouville problems and a single ordinary differential equation.

$$X''(x) + \alpha X(x) = 0, \quad 0 < x < a$$

$$X(0) = 0$$

$$X(a) = 0.$$

$$Y''(y) + \alpha Y(y) = 0, \quad 0 < y < b$$

$$Y(0) = 0$$

$$Y(b) = 0.$$

$$Z''(z) + \alpha Z(z) = 0, \quad 0 < z < c$$

$$Z(0) = 0$$

$$Z(c) = 0.$$

$$T'(t) + (D\lambda - S)T(t) = 0.$$

3.2 Step 2: solve the equations

Here are the solutions to the Sturm-Liouville problems and the ordinary differential equation which were found in class.

$$\alpha_n = \left(\frac{n\pi}{a}\right)^2$$

$$X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, \dots$$

$$\begin{aligned}
\beta_m &= \left(\frac{m\pi}{b}\right)^2 \\
Y_m(y) &= \sin\left(\frac{m\pi y}{b}\right), \quad m = 1, 2, \dots \\
\delta_k &= \left(\frac{k\pi}{c}\right)^2 \\
Z_k(z) &= \sin\left(\frac{k\pi z}{c}\right), \quad k = 1, 2, \dots \\
\lambda_{nmk} &= \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{k\pi}{c}\right)^2 \\
T_{nmk} &= e^{(S-D\lambda_{nmk})t}.
\end{aligned}$$

3.3 Step 3: form the trial solution and find the coefficients

The trial solution is an infinite linear combination of all the solutions found of form (1).

$$\phi(x, y, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} c_{nmk} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{k\pi z}{c}\right) e^{(S-D\lambda_{nmk})t} \quad (9)$$

Using the orthogonality of the eigenfunctions of the Sturm-Liouville problems, the coefficients are given by

$$c_{nmk} = \frac{\int_0^a \int_0^b \int_0^c f(x, y, z) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{k\pi z}{c}\right) dx dy dz}{\int_0^a \int_0^b \int_0^c \sin^2\left(\frac{n\pi x}{a}\right) \sin^2\left(\frac{m\pi y}{b}\right) \sin^2\left(\frac{k\pi z}{c}\right) dx dy dz}. \quad (10)$$

The triple integral in the denominator of (10) can be written as the product of three single integrals, each of which we have already encountered. Our final formula for the coefficients is

$$c_{nmk} = \left(\frac{8}{abc}\right) \int_0^a \int_0^b \int_0^c f(x, y, z) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{k\pi z}{c}\right) dx dy dz. \quad (11)$$

4 Fermi's Analysis

The dominant term in the solution is the one corresponding to $n = 1$, $m = 1$, and $k = 1$,

$$c_{111} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) \sin\left(\frac{\pi z}{c}\right) e^{(S-D\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2\right])t} \quad (12)$$

The dominant term either grows faster or decays slower than all other terms. If the equation

$$S - D \left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2 + \left(\frac{\pi}{c}\right)^2 \right] = 0 \quad (13)$$

is satisfied, then the dominant term will not grow or decay and the flux ϕ will reach a steady state after the other terms have decayed away. The expression on the left of equation (13) is known as the buckling and this equation is known as the buckling equation. It relates material properties, S , D , to geometric properties, a , b , c . Fermi used the buckling equation to estimate the dimensions of a reactor that would allow a self-sustaining (steady state) reaction.

If the buckling is positive, the flux ϕ will grow. If the buckling is much greater than zero, an explosion will occur. If the buckling is negative, the flux ϕ will decay.

Fermi needed a way to control the buckling. He used cadmium control rods, which are neutron absorbers, to control the value of S . He began with the rods inserted all the way into the reactor which produced a negative buckling. He carefully moved the rods out of the reactor, until the buckling increased to zero. He had his self-sustaining reaction.

The Italian navigator had landed.