

Fluxons in dual thin films*

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(Received 28 August 1974)

The field and current distribution for a fluxon penetrating two thin parallel films is obtained. This result is used to calculate the coupling force for an isolated fluxon extending through the two films. The calculation is valid for arbitrary separation between the films when $d_1 \ll \lambda_1$, $d_2 \ll \lambda_2$, and the fluxon core radii are much less than λ . Here d_1 and d_2 are the film thicknesses and λ_1 and λ_2 are the penetration depths. The conditions of validity of a previous calculation for zero separation between the films are established and the comparison with recent experimental work is discussed.

The superconducting dc transformer¹ operates because of the coupling force between fluxons in adjacent thin superconducting films. In a previous paper one of the authors² has calculated this coupling force in the limit that the two films are not separated, using the Pearl model for isolated fluxons³; i.e., both coherence length and film thickness are small compared to the penetration depth. The purpose of this paper is to calculate in general the field and current distribution of a fluxon penetrating a dual thin-film system and to extend the work of Ref. 2 to include the situation where the films are separated by an insulating layer of arbitrary thickness. The present results strongly restrict the range of validity of the results of Ref. 2.

The coupling force which tends to align a fluxon in adjacent films is the negative derivative of the Helmholtz free energy with respect to the lateral distance between fluxon cores.^{2,3} The Helmholtz free energy is given by

$$U = \frac{1}{2} \int \vec{J} \cdot \vec{A} d\tau + (\frac{1}{2} \mu_0 \lambda^2) \int J^2 d\tau, \quad (1)$$

where \vec{J} and \vec{A} are the current density and vector potential, respectively, and the second term is the kinetic energy of the superconducting electrons. For the problem at hand the current and vector potential correspond to the situation shown in Fig. 1, where the fluxon cores are displaced by a distance R parallel to the films. To obtain \vec{J} and \vec{A} we make use of superposition of fluxons, which is valid as long as the normal cores are too small to interfere with the free flow of currents. We write

$$\vec{A} = \vec{A}_1(\vec{r}_1, z) + \vec{A}_2(\vec{r}_2, z), \quad \vec{J} = \vec{J}_1(\vec{r}_1, z) + \vec{J}_2(\vec{r}_2, z) \quad (2)$$

where $\vec{A}_1(\vec{r}_1, z)$ is the vector potential of an isolated fluxon located at $r_1=0$ in the film at $z=a$, with its corresponding screening currents in the film at $z=0$. In the thin-film approximation both \vec{J} and \vec{A} are uniform through the film thickness. If the film at $z=a$ has a penetration depth λ_1 and

thickness d_1 , and the film at $z=0$ has parameters λ_2 , d_2 , then the differential equation obeyed by $\vec{A}_1(\vec{r}_1, z)$ is obtained from the fluxoid quantum conditions:

$$\begin{aligned} \mu_0 \lambda_1^2 \vec{J}_1(\vec{r}_1, a) &= \hat{\phi}_1 (\phi_0 / 2\pi r_1) - \vec{A}_1(\vec{r}_1, a), \\ \mu_0 \lambda_2^2 \vec{J}_1(\vec{r}_1, 0) &= -\vec{A}_1(\vec{r}_1, 0); \end{aligned} \quad (3)$$

$$\nabla \times (\nabla \times \vec{A}_1) = \mu_0 \vec{J}_1:$$

$$\begin{aligned} \nabla_1^2 \vec{A}_1(\vec{r}_1, z) &= (2/\alpha_1) [-\hat{\phi}_1 (\phi_0 / 2\pi r_1) + \vec{A}_1(\vec{r}_1, z)] \\ &\times \delta(z-a) + (2/\alpha_2) \vec{A}_1(\vec{r}_1, z) \delta(z), \end{aligned} \quad (4)$$

where $\alpha_{1,2} = 2\lambda_{1,2}^2/d_{1,2}$ and ϕ_0 is the flux quantum. Similarly, $\vec{A}_2(\vec{r}_2, z)$ is the vector potential for an isolated fluxon located at $r_2=0$ in the film at $z=0$, and is given by the differential equation

$$\begin{aligned} \nabla_2^2 \vec{A}_2(\vec{r}_2, z) &= (2/\alpha_1) \vec{A}_2(\vec{r}_2, z) \delta(z-a) \\ &+ (2/\alpha_2) [-\hat{\phi}_2 (\phi_0 / 2\pi r_2) + \vec{A}_2(\vec{r}_2, z)] \delta(z). \end{aligned} \quad (5)$$

\vec{A}_1 and \vec{A}_2 are azimuthally symmetric about their respective origins, and the corresponding current density \vec{J}_1 is obtained from Eqs. (3) and \vec{J}_2 is obtained from similar equations. The interaction energy of the two fluxons U_{12} is obtained from Eq. (1) with the help of Eqs. (2) and (3) and can be written in the form

$$\begin{aligned} U_{12} &= -\frac{\phi_0}{2\pi\mu_0\alpha_1} \int \frac{da_1 \hat{\phi}_1 \cdot \vec{A}_2(\vec{r}_2, a)}{r_1} \\ &\quad - \frac{\phi_0}{2\pi\mu_0\alpha_2} \int \frac{da_2 \hat{\phi}_2 \cdot \vec{A}_1(\vec{r}_1, 0)}{r_2} \\ &= -\frac{\phi_0}{\mu_0\alpha_1} \int_R^\infty A_2(r_2, a) dr_2 - \frac{\phi_0}{\mu_0\alpha_2} \\ &\quad \times \int_R^\infty A_1(r_1, 0) dr_1. \end{aligned} \quad (6)$$

The force of attraction is then given by $-dU_{12}/dR$,

$$F_c = \frac{-\phi_0 A_2(R, a)}{\mu_0 \alpha_1} - \frac{\phi_0 A_1(R, 0)}{\mu_0 \alpha_2}. \quad (7)$$

Later it will be seen that the above two terms are

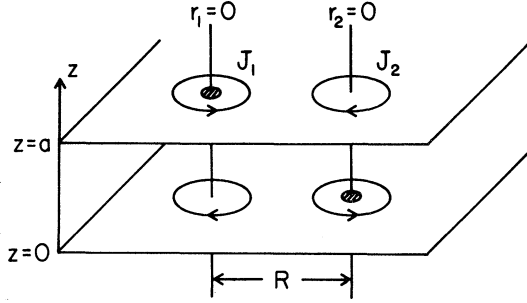


FIG. 1. Fluxon penetrating two parallel thin films separated by an insulating layer of thickness a with a lateral separation R between the two cores. J_1 is the current density of an isolated vortex centered at $r_1=0$ in the film at $z=a$, with its corresponding screening current in the film at $z=0$. Similarly, J_2 is the current density for an isolated vortex centered at $r_2=0$ in the film at $z=0$, with screening currents in the film at $z=a$.

equal and with the use of Eqs. (3) one can cast Eq. (7) into a form which is very similar to the repulsive force between two fluxons in a single film; i.e., the force is proportional to the screening currents of one vortex evaluated at the core of the other,

$$F_c = \phi_0 d_1 J_2(R, a) = \phi_0 d_2 J_1(R, 0). \quad (8)$$

However, in this case the current is proportional to the vector potential, leading to a $1/R$ asymptotic dependence as opposed to the $1/R^2$ dependence obtained for two fluxons in a single film.

To calculate the force of Eq. (7) it is necessary to solve Eq. (4) or Eq. (5) for the vector potential of a pair of parallel films with a fluxon penetrating one of them, and, in fact, it is necessary to know only the vector potential in the film carrying screening currents. Since this situation has azimuthal symmetry, the vector potential has only a ϕ component, and Eq. (5) reduces to

$$\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} (rA) + \frac{\partial^2}{\partial z^2} A = \frac{2}{\alpha_1} A \delta(z-a) + \frac{2}{\alpha_2} \left(-\frac{\phi_0}{2\pi r} + A \right) \delta(z), \quad (9)$$

where for convenience we have dropped unnecessary subscripts. This equation is separable and we write the solution as the product of a Fourier transform in z and a Hankel transform in r , which leads to a solution of the form

$$A(r, z) = \frac{\phi_0}{2\pi} \int_0^\infty dp J_1(pr) M(p, z), \quad (10)$$

where J_1 is the Bessel function of order unity and

$$M(p, z) = \frac{(1+p\alpha_1)e^{-p|z|} - e^{-p|z-a|}e^{-pa}}{(1+p\alpha_1)(1+p\alpha_2) - e^{-2pa}}. \quad (11)$$

When Eq. (10) is evaluated for $A_2(R, a)$, it is found to be equal to the product of α_1 and a function

symmetric in α_1 and α_2 . Similarly, $A_1(R, 0)$ is equal to α_2 multiplied by the same symmetric function. When these results are inserted in Eq. (7), the factors of $\alpha_{1,2}$ cancel, the two terms are clearly seen to be equal, and the force takes the final form

$$F_c = -\frac{\phi_0^2}{\pi\mu_0} \int_0^\infty \frac{dp p e^{-pa} J_1(pR)}{(1+p\alpha_1)(1+p\alpha_2) - e^{-2pa}} \quad (12)$$

Equation (12) is not readily integrated in terms of well-known functions and, although it could be expressed in terms of series expansions, it is perhaps most easily handled in its present form as an integral transform. A plot of the force versus lateral core separation for typical values of the parameters is shown in Fig. 2. It is seen that the force starts off linearly in R , rises to a maximum, and then goes over into a $1/R$ asymptotic behavior.

Although Eq. (12) cannot be readily integrated, we can easily determine some important limiting cases. If either $\alpha_1 \gg a$ or $\alpha_2 \gg a$, we have

$$F_c = -\frac{2\phi_0}{\mu_0} \frac{1}{\alpha_1 + \alpha_2} A_{\alpha'}(R, a), \quad (13)$$

where $\alpha' = \alpha_1\alpha_2/(\alpha_1 + \alpha_2)$ and $A_{\alpha'}(r, z)$ is the vector potential of an isolated fluxon in a single film of effective parameter α' calculated in the Pearl model:

$$A_{\alpha'}(r, z) = \frac{\phi_0}{2\pi} \int_0^\infty dp \frac{e^{-pz} J_1(pr)}{1 + \alpha' p}. \quad (14)$$

If in addition one of the films is very nearly diamagnetic with respect to the other, i.e., if $\alpha_1 \gg a \gg \alpha_2$ or $\alpha_2 \gg a \gg \alpha_1$, Eqs. (13) and (14) can be evaluated to obtain

$$F_c = -\frac{\phi_0^2}{\pi\mu_0} \frac{1}{\alpha_1 + \alpha_2} \frac{1}{R} \left(1 - \frac{a}{(a^2 + R^2)^{1/2}} \right). \quad (15)$$

In the limit that the separation between films is large, $a \gg \alpha_1, \alpha_2$, the force can be expanded for

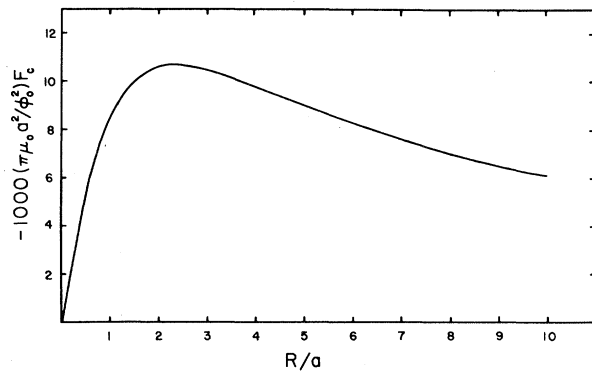


FIG. 2. Coupling force as a function of lateral core separation. Here $-(\pi\mu_0 a^2/\phi_0^2)F_c$ is plotted vs R/a for the case $\alpha_1 = \alpha_2 = \alpha$ and $a/\alpha = 0.2$.

small R as

$$F_c = -\frac{\phi_0^2}{\pi\mu_0} \frac{R}{a^3} \left(\frac{7}{8} \zeta(3) - \frac{R^2}{a^2} \frac{93}{64} \zeta(5) + \dots \right), \quad (16)$$

where $\zeta(3) = 1.20205$ and $\zeta(5) = 1.03693$ are Riemann ζ functions. The asymptotic form of Eq. (12) is obtained upon expanding the integrand in powers of a/R and is

$$F_c = \frac{\phi_0^2}{\pi\mu_0} \frac{1}{(\alpha_1 + \alpha_2 + 2a)} \frac{1}{R}; \quad R \gg a, \quad R \gg \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2 + 2a}. \quad (17)$$

Of special interest is the case of zero separation between the films. Setting $a = 0$, the force of Eq. (12) becomes

$$F_c = -\frac{2\phi_0}{\mu_0} \frac{1}{(\alpha_1 + \alpha_2)} A_{\alpha'}(R, 0), \quad (18)$$

where $A_{\alpha'}(R, 0)$, the solution for the vector potential of Eq. (14) in the plane of the film, is shown by Pearl to be

$$A_{\alpha'}(R, 0) = \frac{\phi_0}{2\pi} \left\{ \frac{1}{R} - \frac{\pi}{2\alpha'} \left[S_1\left(\frac{R}{\alpha'}\right) - N_1\left(\frac{R}{\alpha'}\right) - \frac{2}{\pi} \right] \right\}. \quad (19)$$

S_1 and N_1 are, respectively, the Struve and Neumann functions. Equation (18) is the result obtained in Ref. 2 and, although this has the same asymptotic dependence as Eq. (17), it should be pointed out that it exhibits unphysical behavior in the region of small R . For $R \ll \alpha'$ Eq. (19) approaches the finite value $\phi_0/2\pi\alpha'$ (implying an infinite magnetic field) instead of the linear behavior in R given by the general solution (12) for $a \neq 0$. This is a direct result of the assumption of zero coherence length in the Pearl model and would not occur if the core were of finite size. In practice the films are of finite thickness with nonzero coherence length and are separated by an insulating layer, and any one of these properties taken independently will ensure that the force will have the type of behavior for small R which is shown in Fig. 2.

It is the maximum value of the coupling force which can be determined experimentally.⁴⁻⁶ The force F_c of Eq. (12), however, probably overestimates the maximum coupling which can exist in a transformer, since normally the fluxons are not isolated and the overlap of their current and field distributions serves to decrease the coupling. Equation (12) may represent the transformer cou-

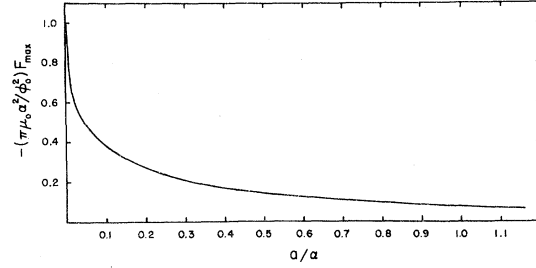


FIG. 3. Maximum coupling force as a function of film parameters for the situation in which $\alpha_1 = \alpha_2 = \alpha$. The scaled maximum force $-(\pi\mu_0\alpha^2/\phi_0^2)F_{\max}$ is plotted as a function of a/α .

pling reasonably well in situations where there is no applied magnetic field, the thin-film approximation $d \ll \lambda$ is valid, and the fluxon cores are small. Clem has recently developed a dynamical model of coupled fluxon arrays which allows him to calculate I - V characteristics, which are in good agreement with measurements on a transformer with a 12-nm separation between granular aluminum films.⁴ The maximum coupling force calculated here appears as one of five parameters in his model.

The maximum value of the calculated coupling force is shown in Fig. 3 as a function of a/α for the case $\alpha_1 = \alpha_2 = \alpha$. It is seen that $-(\pi\mu_0\alpha^2/\phi_0^2)F_{\max}$ has a maximum of unity at $a/\alpha = 0$ and falls off very rapidly for small values of a/α , while for values of a/α of order unity it is a relatively slowly varying function. This large initial value and extremely sharp decrease is again a direct result of assuming zero coherence length in the model and implies that this calculation is not physically significant for film separation less than the coherence length. In general, the calculations indicate that the coupling should decrease with increasing a , and for $a/\alpha \sim 0.1$ should also decrease for increasing α .

Measurements showing a decrease in the coupling force with increasing separation a and increasing α have been reported recently for tin transformers.^{6,7} These tin films do not, however, correspond to the thin-film limit discussed here, because for them the coherence length is always greater than λ and d is sometimes greater than λ . The present calculation should be most directly applicable to thin type-II films in zero magnetic field.

*Research supported in part by the National Science Foundation.

¹I. Giaever, Phys. Rev. Lett. **15**, 825 (1965); **16**, 460

(1966); P. R. Solomon, *ibid.* **16**, 50 (1966); R. Deltour and M. Tinkham, Phys. Rev. **174**, 478 (1968); P. E. Cladis, Phys. Rev. Lett. **21**, 1238 (1968); P. E. Cladis,

- R. D. Parks, and J. M. Daniels, *ibid.* 21, 1521 (1968).
- ²Max D. Sherrill, Phys. Rev. B 7, 1908 (1973).
- ³J. Pearl, Appl. Phys. Lett. 16, 50 (1966); in *Proceedings of the Ninth International Conference on Low Temperature Physics* (Plenum, New York, 1965), Part A, p. 566.
- ⁴John R. Clem, Phys. Rev. B 9, 898 (1974).
- ⁵J. W. Ekin, B. Serin, and John R. Clem, Phys. Rev. B 9, 912 (1974).
- ⁶Max D. Sherrill and William A. Lindstrom, Phys. Rev. B 11, 1125 (1975).
- ⁷W. A. Lindstrom and M. D. Sherrill, Phys. Lett. A 46, 77 (1973).