1 Introduction

This chapter is an introduction to hyperquadrics. Since the hyperquadric is a generalization of the superquadric, we start with a brief overview of superquadrics and then extend those ideas to hyperquadrics. While the algebra is presented rigorously, the emphasis is on the geometry. It is the geometric understanding of the hyperquadric equation that helps us design new algorithms.

2 Superquadrics

Superquadrics were first discovered by the Danish designer Piet Hein (see Gardner [?]) and introduced in the computer graphics literature by Barr [?, ?]. A superquadric is essentially a generalized ellipsoid whose exponents are allowed to take arbitrary real values. A superquadric in 3D is the set of points \([x, y, z]^T\) that satisfy the following equation.

\[
\left| \frac{x}{a} \right|^\gamma_1 + \left| \frac{y}{b} \right|^\gamma_2 + \left| \frac{z}{c} \right|^\gamma_3 = 1 \quad \gamma_i > 0 \quad \forall i = 1, 2, 3
\]

Note that if we set the \(\gamma_i\)'s to 2, the above equation will represent an ellipse.

To get a geometric understanding of the above equation, note that each term in the above summation is non-negative and these terms add up to one. This means that each term is less than or equal to one. i.e.

\[
\left| \frac{x}{a} \right|^\gamma_1 \leq 1
\]

\[
\left| \frac{y}{b} \right|^\gamma_2 \leq 1
\]

\[
\left| \frac{z}{c} \right|^\gamma_3 \leq 1
\]

Since the \(\gamma_i\)'s are positive, the above set of equations imply the following.

\[
\left| \frac{x}{a} \right| \leq 1 \quad \Rightarrow \quad -a \leq x \leq +a
\]

\[
\left| \frac{y}{b} \right| \leq 1 \quad \Rightarrow \quad -b \leq y \leq +b
\]

\[
\left| \frac{z}{c} \right| \leq 1 \quad \Rightarrow \quad -c \leq z \leq +c
\]

Thus, the set of points \([x, y, z]^T\) lying on the superquadric satisfy \(-a \leq x \leq +a, -b \leq y \leq +b\) and \(-c \leq z \leq +c\). In other words, the superquadric is bounded by the three pairs of planes given by the following equations.

\[
x = \pm a
\]

\[
y = \pm b
\]

\[
z = \pm c
\]
Note that each equation above represents a pair parallel planes. The above planes enclose a closed volume which contains the superquadric. These planes are therefore called the bounding planes. When the exponents ($\gamma_i$'s) are all equal to 2, we have an ellipsoid bound by the bounding planes. As the exponents become larger and larger the shape of the curve approaches that of the bounding box. When the exponents become less than 1, we get concave shapes.

Figures 1 through 5 show some superquadrics in 2D. Superquadrics in 2D are similar to those in 3D except that they do not have the z component. In 2D, we have bounding lines instead of bounding planes. The caption below each of the following figures show the values of the parameters. Each figure also shows the bounding boxes. In figure 1, for example, the two vertical lines correspond to $x = \pm 200$ and the horizontal lines correspond to $y = \pm 100$. The parameters $a$ and $b$ determine the “size” of the bounding box and the parameters $\gamma_1$ and $\gamma_2$ determine the “shape” of the curve. As can be seen from the figures, as the $\gamma_i$'s tend to infinity, the curve approaches the bounding box. As the $\gamma_i$'s tend to zero, the shape becomes more and more concave and “pinched”. Figures 6 through 8 show some superquadrics in 3D.

Figure 1: Superquadric in 2D. $a = 200$ $b = 100$ $\gamma_1 = 2$ $\gamma_2 = 2$

Figure 2: Superquadric in 2D. $a = 200$ $b = 100$ $\gamma_1 = 7$ $\gamma_2 = 2$
Figure 3: Superquadric in 2D. \( a = 200 \ b = 100 \ \gamma_1 = 1 \ gamma_2 = 2 \)

Figure 4: Superquadric in 2D. \( a = 200 \ b = 100 \ \gamma_1 = 0.5 \ gamma_2 = 2 \)

Figure 5: Superquadric in 2D. \( a = 200 \ b = 20 \ \gamma_1 = 2 \ gamma_2 = 2 \)
Figure 6: Superquadric in 3D as seen from an arbitrary point in space

Figure 7: Superquadric in 3D as seen from an arbitrary point in space
Figure 8: Superquadric in 3D as seen from an arbitrary point in space.
As can be seen from the above figures, the superquadric can model a diverse range of shapes. But still, they have one major limitation - superquadrics are inherently symmetric about the three coordinate planes. This is due to the absolute value operators in the superquadric equation. As can be easily seen from the equation, if a point with \([x, y, z]^T\) lies on the superquadric so does the point \([-x, y, z]^T\). This means that the shape is symmetric about the Y-Z plane. Similar arguments show that the shape is symmetric about the other two coordinate planes as well.

The superquadric equation can be expressed in parametric form in terms of the angles \(\theta\) and \(\phi\). The parametric equations are

\[
\begin{align*}
x &= a \text{ sign}(\cos(\theta)\cos(\phi)) |\cos(\theta)|^{\alpha_1} |\cos(\phi)|^{\alpha_2} \\
y &= b \text{ sign}(\sin(\theta)\cos(\phi)) |\sin(\theta)|^{\alpha_1} |\cos(\phi)|^{\alpha_2} \\
z &= c \text{ sign}(\sin(\theta)) |\sin(\phi)|^{\alpha_3}
\end{align*}
\]  

(11)  
(12)  
(13)

where \(\alpha_i = \gamma_i \quad \forall i = 1, 2, 3, -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\) and \(-\pi \leq \theta \leq \pi\). In other words, as \(\theta\) and \(\phi\) vary with in their limits the tip of the vector \([x, y, z]^T\) traces the surface of the superquadric. The above parametric form is very useful while fitting superquadrics to data.

While the superquadric can generate a wide range of shapes, its symmetry is a severe limitation of its modeling capabilities. Many natural and man-made objects are asymmetric and hence can not be modeled by the superquadric. To alleviate this problem, Barr [?] introduced global and local deformations to superquadrics. He introduced many different types of deformations like bending, tapering etc. Deformable superquadrics can model a wider range of shapes but fitting such shapes to real data is usually difficult except for simple deformations.

In the computer vision literature, the following variation of the superquadric equation has been traditionally used [?].

\[
\left( \frac{x}{a_1} \right)^{\frac{2}{c_1}} + \left( \frac{y}{a_2} \right)^{\frac{2}{c_2}} + \left( \frac{z}{a_3} \right)^{\frac{2}{c_3}} = 1
\]  

(14)

A number of algorithms have been proposed for fitting the above superquadric to real data (see references given above). All these algorithms first define an error function that measures the distance between the model and the data and minimize the error function.

With the above background on superquadrics, let us now turn to a discussion of hyperquadrics.

3 Hyperquadrics

Hyperquadrics are parametric shape models that are more powerful than superquadrics and include superquadrics as a special case. They were first proposed for computer graphics applications by Hanson [?]. Hyperquadrics can model shapes that are not necessarily symmetric and their geometric bounds are arbitrary convex polyhedra as opposed to superquadrics whose geometric bounds are simple Cartesian rectangular parallelopipeds. (More complex geometric bounds can also be obtained for hyperquadrics as suggested in [?].) The power of
hyperquadrics arises from the fact that they can model some shapes that cannot be described by deformable superquadrics.

A hyperquadric is a surface defined by the set of points $(x, y, z)$ satisfying the following equation.

$$\sum_{i=1}^{N} | A_i x + B_i y + C_i z + D_i |^{\gamma_i} = 1$$ (15)

Here $N$ is any arbitrary number and $\gamma_i \geq 0 \forall i$. We note that a superquadric as defined by

$$| \frac{x}{a} |^{\gamma_1} + | \frac{y}{b} |^{\gamma_2} + | \frac{z}{c} |^{\gamma_3} = 1$$ (16)

is a special case of the hyperquadric with $N = 3$, $A_1 = \frac{1}{a}$, $B_2 = \frac{1}{b}$, $C_3 = \frac{1}{c}$ and all other parameters (except $\gamma_i$) zero. Whereas the superquadric has only 3 terms, the hyperquadric has an arbitrary number of terms. The hyperquadric can model a much broader range of shapes than the superquadric.

To get a geometric feeling for the hyperquadric equation, note that each term in the summation is positive and all the terms add up to one. Therefore, each of the individual terms can never exceed one. \textit{i.e.},

$$| A_i x + B_i y + C_i z + D_i |^{\gamma_i} \leq 1 \forall i$$ (17)

Since $\gamma_i$’s are positive, the above inequalities imply that

$$| A_i x + B_i y + C_i z + D_i | \leq 1 \forall i = 1, 2, \ldots, N$$ (18)

If we replace the inequalities by equalities, then the above equations describe a pair of parallel planes for each $i$. The intersection of all these 2$N$ planes defines a convex polytope and the hyperquadric is constrained to lie within this polytope. We therefore call these planes bounding planes. More complex bounding volumes can be defined by using more complex expressions within each term [9]. For example, if the planar expression $A_i x + B_i y + C_i z + D_i$ is replaced with an expression for a sphere, the resulting bounding volume would be the intersection of a set of spheres.

To understand the effect of the various parameters on the resulting shape let us consider the 2-D hyperquadric:\footnote{We use 2-D hyperquadrics for illustration purposes.}

$$\sum_{i=1}^{N} | A_i x + B_i y + D_i |^{\gamma_i} = 1$$ (19)

Instead of bounding planes, we now have bounding lines. Note that each term gives a pair of parallel lines. The parameters $A_i$ and $B_i$ in term $i$ give the slope of the lines corresponding to that term. The distance $S_i$ between the lines is given by $\frac{2}{\sqrt{A_i^2 + B_i^2}}$. One line is at a distance $\frac{D_i - 1}{\sqrt{A_i^2 + B_i^2}}$ from the origin and the other is at a distance $\frac{D_i + 1}{\sqrt{A_i^2 + B_i^2}}$. The distance $T_i$ between the
origin and the center-plane between the two bounding planes is given by $T_i = \frac{D_i}{2s}$. Figure 9 shows four hyperquadrics, each with 2 terms. The bounding lines are also shown. If a particular value of $\gamma$ is high, the shape is pulled towards the corresponding pair of bounding lines. When all the $\gamma$'s are 2, the resulting shape is an ellipse. When the $\gamma$'s are 1, the shape is a rectangle and when a $\gamma$ becomes less than one, the shape becomes concave. Figures 10 through 12 show some 3D hyperquadrics. Each of them has 4 terms in its equation.
Figure 9: 2D Hyperquadrics: Inclined lines correspond to term 1; horizontal lines to term 2.
An attractive property of the hyperquadrics is that they are not necessarily symmetric. Also, since the number of terms \( N \) is arbitrary, we can add more and more terms to the equation to model a given shape closely. Further, if a single parameter is varied slightly, the shape changes only slightly in the direction perpendicular to the corresponding bounding planes. This is in contrast to spherical harmonics.
where, due to global basis functions, a change in one parameter affects the whole shape considerably.

While the hyperquadric has a number of attractive properties, a significant drawback as far as computer vision is concerned is that no known parametric form exists for the hyperquadric equation. Superquadrics, on the other hand can be expressed in the parametric form as given by equations 11 thorough 13. The parametric form of the superquadric is useful for both displaying the shape (graphics applications) and in designing good fitting algorithms (computer vision applications). Hanson [?] describes a numerical approach for solving the hyperquadric equation for display purposes. This method uses radial parameterization with Newton’s method to determine good numerical solutions. To avoid points of inflections he uses the “fail-safe” Newton’s method [?].

To avoid the difficulties associated with the Newton’s method, we used an less efficient but stable algorithm for finding a set of points that lie on a given hyperquadric (for display purposes). We use a spherical coordinate system and for each choice of \( \theta \) and \( \phi \), we determine \( r \) such that the point with coordinates \((r, \theta, \phi)\) lies on the hyperquadric. To determine the correct \( r \), we vary \( r \) between a lower limit and an upper limit and at each value we check the Inside-Outside (IO) function (to be discussed in the next chapter) to determine if the current value gives a point on (or sufficiently) close to the hyperquadric. The IO function is positive for points outside the hyperquadric and negative for points inside the hyperquadric. Therefore, the zero-crossing of the IO function as a function of \( r \) gives us an estimate of the correct \( r \). Upper and lower limits on \( r \) can be easily determined by analyzing the bounding box. All figures shown in this dissertation were obtained this way.

4 Conclusions

This chapter presented an introduction to hyperquadrics. The modeling power of the hyperquadric is rather obvious. A major attractive feature is the intuitive geometry behind the hyperquadric equation. When a given shape is not modeled adequately by a hyperquadric with certain number of terms, new terms can be easily added to “trim” the model. All the parameters have intuitive meaning and predictable changes to shape can be obtained by changing these parameters. Superquadrics share some of the properties of hyperquadrics
but their symmetry is a major limitation. Superquadrics model only a subset of the shapes that can be modeled by the hyperquadric. In the next chapter we will discuss the problem of fitting hyperquadrics to data points. There, we will study some of the numerical difficulties that are unique to hyperquadrics and study methods to overcome those difficulties.

![Image of a hyperquadric in 3D](image)

**Figure 12** A hyperquadric in 3D

**Conclusions**

The chapter presents an introduction to hyperquadrics. The motivation power of the hyperquadric is rather obvious. A major attraction is its intuitive geometric meaning and the geometric properties that are unique to hyperquadrics. Superquadrics are defined on a non-linear manifold and their geometric properties are not as straightforward as those of hyperquadrics. However, certain numerical issues can arise in dealing with hyperquadrics. Superquadrics span some of the properties of hyperquadrics.