Exercise H7.2 Let $X_1, \ldots, X_{n_1}$, be independent $N(\mu_1, \sigma^2)$ and $Y_1, \ldots, Y_{n_2}$ be independent $N(\mu_2, \sigma^2)$, also independent of $X_1, \ldots, X_{n_1}$ (n_1, $n_2 > 1$). In a model where these r.v.'s are observed, and μ_1, μ_2 and σ^2 are all unknown, find an α-test for the hypothesis

$$H : \mu_1 = \mu_2.$$

Hint: compare exercise H6.1, previous homework.

Sollution: in the exercise H6.1, we know

$$\mathcal{L}(T(X,Y)) = t\,(n_1 + n_2 - 2)\,,$$

where

$$T(X,Y) = \frac{(\overline{X} - \overline{Y})\,(n_1 + n_2 - 2)^{\frac{1}{2}}}{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{\frac{1}{2}} \left((n_1 - 1)\widehat{S}^2_{(1)} + (n_2 - 1)\widehat{S}^2_{(2)}\right)^{\frac{1}{2}}}.$$

Let $z_{\alpha/2, n_1+n_2-2}$ be the upper $\alpha/2-$quantile of the $t-$distribution for $n_1 + n_2 - 2$ degrees of freedom, and

$$\phi(X,Y) = 1_A\,(T(X,Y))\,,$$

where $A = \{x \in R : |x| \geqslant z_{\alpha/2, n_1+n_2-2}\}$,

then $\phi(X,Y)$ is the α-test for the hypothesis $H : \mu_1 = \mu_2.$

Exercise H7.3 Let $z_{\alpha/2,n}$ be the upper $\alpha/2$-quantile of the t-distribution with n degrees of freedom and $z^*_{\alpha/2}$ the respective quantile for the standard normal distribution. Use the tables at the end of the textbook or a computer program to find
a) $z_{\alpha/2,n}$ for $n = 5$, $n = 20$ and $z^*_{\alpha/2}$ for a value $\alpha = 0.05$
b) the same for $\alpha = 0.01$.

Solution: we have the following values:

α	$z_{\alpha/2,5}$	$z_{\alpha/2,20}$	$z^*_{\alpha/2}$
0.01	4.032142984	2.845339710	2.575829304
0.05	2.570581836	2.085963447	1.959963985

Let $n \to \infty$, what conclusion you can get? Try to prove it.