HW #14. (Mthsc 206)

1. $z = x^2 + y^2 + xy,\ x = \sin t,\ y = e^t \ \Rightarrow\ \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt} = (2x + y)\cos t + (2y + x)e^t$

2. $z = \cos(x + 4y),\ x = 5t^4,\ y = 1/t \ \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = -\sin(x + 4y)(1)(20t^3) + [-\sin(x + 4y)(4)](-t^{-2})$$

$$= -20t^3\sin(x + 4y) + \frac{4}{t^2}\sin(x + 4y) = \left(\frac{4}{t^2} - 20t^3\right)\sin(x + 4y)$$

3. $z = \sqrt{1 + x^2 + y^2},\ \ x = \ln t,\ \ y = \cos t \ \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(1+x^2+y^2)^{-1/2}(2x)\cdot\frac{1}{t} + \tfrac{1}{2}(1+x^2+y^2)^{-1/2}(2y)(-\sin t) = \frac{1}{\sqrt{1+x^2+y^2}}\left(\frac{x}{t} - y\sin t\right)$$

7. $z = x^2y^3,\ x = s\cos t,\ y = s\sin t \ \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = 2xy^3\cos t + 3x^2y^2\sin t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2xy^3)(-s\sin t) + (3x^2y^2)(s\cos t) = -2sxy^3\sin t + 3sx^2y^2\cos t$$

8. $z = \arcsin(x - y),\ \ x = s^2 + t^2,\ \ y = 1 - 2st \ \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{\sqrt{1-(x-y)^2}}(1)\cdot 2s + \frac{1}{\sqrt{1-(x-y)^2}}(-1)\cdot(-2t) = \frac{2s+2t}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{\sqrt{1-(x-y)^2}}(1)\cdot 2t + \frac{1}{\sqrt{1-(x-y)^2}}(-1)\cdot(-2s) = \frac{2s+2t}{\sqrt{1-(x-y)^2}}$$

9. $z = \sin\theta\cos\phi,\ \ \theta = st^2,\ \ \phi = s^2t \ \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial s} = (\cos\theta\,\cos\phi)(t^2) + (-\sin\theta\,\sin\phi)(2st) = t^2\cos\theta\,\cos\phi - 2st\sin\theta\,\sin\phi$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi}\frac{\partial \phi}{\partial t} = (\cos\theta\,\cos\phi)(2st) + (-\sin\theta\,\sin\phi)(s^2) = 2st\cos\theta\,\cos\phi - s^2\sin\theta\,\sin\phi$$

21. $z = x^2 + xy^3,\ x = uv^2 + w^3,\ y = u + ve^w \ \Rightarrow$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1),$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w),$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w).$$

When $u = 2$, $v = 1$, and $w = 0$, we have $x = 2$, $y = 3$,

so $\dfrac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85,\ \dfrac{\partial z}{\partial v} = (31)(4) + (54)(1) = 178,\ \dfrac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54.$

22. $u = (r^2 + s^2)^{1/2}$, $r = y + x\cos t$, $s = x + y\sin t \Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \tfrac{1}{2}(r^2+s^2)^{-1/2}(2r)(\cos t) + \tfrac{1}{2}(r^2+s^2)^{-1/2}(2s)(1) = (r\cos t + s)/\sqrt{r^2+s^2},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \tfrac{1}{2}(r^2+s^2)^{-1/2}(2r)(1) + \tfrac{1}{2}(r^2+s^2)^{-1/2}(2s)(\sin t) = (r + s\sin t)/\sqrt{r^2+s^2},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial t} = \tfrac{1}{2}(r^2+s^2)^{-1/2}(2r)(-x\sin t) + \tfrac{1}{2}(r^2+s^2)^{-1/2}(2s)(y\cos t) = \frac{-rx\sin t + sy\cos t}{\sqrt{r^2+s^2}}.$$

When $x = 1$, $y = 2$, and $t = 0$ we have $r = 3$ and $s = 1$, so $\dfrac{\partial u}{\partial x} = \dfrac{4}{\sqrt{10}}$, $\dfrac{\partial u}{\partial y} = \dfrac{3}{\sqrt{10}}$, and $\dfrac{\partial u}{\partial t} = \dfrac{2}{\sqrt{10}}$.

23. $R = \ln(u^2 + v^2 + w^2)$, $u = x + 2y$, $v = 2x - y$, $w = 2xy \Rightarrow$

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \frac{2u}{u^2+v^2+w^2}(1) + \frac{2v}{u^2+v^2+w^2}(2) + \frac{2w}{u^2+v^2+w^2}(2y)$$
$$= \frac{2u + 4v + 4wy}{u^2+v^2+w^2},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y} = \frac{2u}{u^2+v^2+w^2}(2) + \frac{2v}{u^2+v^2+w^2}(-1) + \frac{2w}{u^2+v^2+w^2}(2x)$$
$$= \frac{4u - 2v + 4wx}{u^2+v^2+w^2}.$$

When $x = y = 1$ we have $u = 3$, $v = 1$, and $w = 2$, so $\dfrac{\partial R}{\partial x} = \dfrac{9}{7}$ and $\dfrac{\partial R}{\partial y} = \dfrac{9}{7}$.