HW #7 (MthSc 400)

n1

3.84 $M(t) = E(e^{tY}) = \sum_{y=0}^{1} e^{ty} p(y) = e^{t(0)}(1-p) + e^{t(1)}p = 1 - p + e^t p$

3.85 $M(t) = E(e^{tY})$

$$= \sum_{y=0}^{n} e^{ty} \binom{n}{y} p^y (1-p)^{(n-y)} = \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y (1-p)^{(n-y)} = [pe^t + (1-p)]$$

since, using the binomial theorem, we have

$$\sum_{x=0}^{n} \binom{n}{x} a^x b^{(n-x)} = (a+b)^n.$$

Therefore,

$$E(Y) = M'(0) = n[pe^t + (1-p)]^{n-1} pe^t \big|_{t=0} = np$$

$$E(Y^2) = M''(0)$$

$$= (n(n-1)[pe^t + (1-p)]^{n-1}(pe^t)^2 + n[pe^t + (1-p)]^{n-1} pe^t)\big|_{t=0}$$

$$= n(n-1)p^2 + np$$

$$V(Y) = E(Y^2) - [E(Y)]^2$$

$$= n(n-1)p^2 + np - (np)^2 = np(1-p)$$

3.86 $M(t) = E(e^{tY})$

$$= \sum_{y=0}^{\infty} e^{ty} \frac{\lambda^y e^{-\lambda}}{y!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!} = e^{-\lambda} e^{\lambda e^t} \left(\text{since } \sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a\right) = e^{\lambda(e^t - 1)}$$

$$E(Y) = M'(0) = e^{\lambda(e^t-1)} \lambda e^t \big|_{t=0} = \lambda$$

$$E(Y^2) = M''(0) = e^{\lambda(e^t-1)} (\lambda e^t)^2 + e^{\lambda(e^t-1)} \lambda e^t \big|_{t=0} = \lambda^2 + \lambda$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

3.87 $M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)}) = E[e^{tb} e^{(at)X}] = e^{tb} E[e^{(at)X}] = e^{tb} M_X(at)$

3.88 $E(Y) = M'_Y(0) = (be^{tb}M_X(at) + e^{tb}M'_X(at)\,a)\big|_{t=0}$

$= bM_X(0) + aE(X)$

$= aE(X) + b$ (since $M_X(0) = E(e^{(0)(X)}) = E(1) = 1$)

$E(Y^2) = M''_Y(0)$

$= (b^2e^{tb}M_X(at) + abe^{tb}M'_X(at) + abe^{tb}M'_X(at) + a^2e^{tb}M''_X(at))\big|_{t=0}$

$= b^2 + 2abE(X) + a^2E(X^2)$

$V(Y) = E(Y^2) - [E(Y)]^2 = b^2 + 2abE(X) + a^2E(X^2) - [aE(X) + b]^2$

$= a^2(E(X^2) - [E(X)]^2) = a^2V(X)$