

Note on the closed-form MLEs of k -component load-sharing systems

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Abstract

Consider a multiple component system connected in parallel. In this system, as components fail one by one, the total load or traffic applied to the system is redistributed among the remaining surviving components, which is commonly referred to as *load-sharing*.

Recently Kim and Kvam (2004) and Singh et al. (2008) proposed different load-sharing models and developed parametric inference for the these models. However, their parametric estimates are calculated using iterative numerical methods. In this note, we provide the general closed-form MLEs for the two load-sharing models provided by them.

1 Introduction

Most research work involving load sharing models has mainly focused on the characterization of system reliability under a known load-sharing rule and parameters. The parameter estimation of the load-sharing rule has not yet been fully developed. Recently, parametric inference for reliability under the equal load-sharing rule has been considered by Kim and Kvam (2004) and Singh et al. (2008). They solved the likelihood estimating equations to

find the maximum likelihood estimators (MLEs) of the load-sharing parameters.

However, they provide no general closed form solutions for the MLEs, but instead use iterative numerical methods to calculate their estimates. It is well known that there are some problematic issues associated with iterative numerical methods such as stability and convergence.

In this note, we provide the general closed-form MLEs for the two load-sharing models provided by Kim and Kvam (2004) and Singh et al. (2008).

2 Kim-Kvam load-sharing model

Consider k -component system connected in parallel. Following Kim and Kvam (2004), we assume the following:

- (i) A system is made up of k components whose lifetimes are independent and have identical exponential distributions with initial failure rate θ .
- (ii) After the first component fails, the failure rates of the remaining $k - 1$ components change to $\lambda_1\theta$ where $\lambda_1 > 0$. After the next component failure, the failure rates of the surviving $k - 2$ components change to $\lambda_2\theta$. Then, after the next component failure, the the failure rates of the surviving $k - 3$ components change to $\lambda_3\theta$ and so on and so forth.
- (iii) There are n repeated measurements of independent systems. That is, we have a random sample of independent systems of size n .

Let T_{ij} denote the time spacing between the $(j - 1)$ -th failure and j -th failure for the i -th system where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, k$. As shown in Kim and Kvam (2004), the likelihood function for the i -th system is

$$L_i(\theta, \Lambda) = (k!)^n \theta^k \cdot \left[\prod_{j=1}^k \lambda_{j-1} \right] \cdot \exp \left[-\theta \sum_{j=1}^k (k - j + 1) \lambda_{j-1} t_{ij} \right],$$

where $\lambda_0 = 1$ and $\Lambda = (\lambda_1, \dots, \lambda_{k-1})$. It is immediate that the likelihood function for a random sample of size n is given by

$$L(\theta, \Lambda) = (k!)^n \theta^{nk} \cdot \left[\prod_{j=1}^k \lambda_{j-1}^n \right] \cdot \exp \left[-\theta \sum_{i=1}^n \sum_{j=1}^k (k - j + 1) \lambda_{j-1} t_{ij} \right]. \quad (1)$$

Taking the logarithm of (1), differentiating with respect to $\theta, \lambda_1, \dots, \lambda_{k-1}$, denoting partial derivative of $\log L$ with respect to θ as $\ell_\theta = \partial \log L / \partial \theta$ and partial derivative of $\log L$ with respect to λ_{j-1} as $\ell_{j-1} = \partial \log L / \partial \lambda_{j-1}$, we obtain the log-likelihood estimating equations shown below:

$$\ell_\theta = \frac{nk}{\theta} - \sum_{i=1}^n \sum_{j=1}^k (k - j + 1) \lambda_{j-1} t_{ij} = 0 \quad (2)$$

and

$$\ell_{j-1} = \frac{n}{\lambda_{j-1}} - \theta \sum_{i=1}^n (k - j + 1) t_{ij} = 0 \quad (3)$$

for $j = 2, 3, \dots, k$. For convenience, we denote $t_{\bullet j} = \sum_{i=1}^n t_{ij}$. Then, (2) and (3) can be rewritten as

$$\ell_\theta = \frac{nk}{\theta} - \sum_{j=1}^k (k - j + 1) \lambda_{j-1} t_{\bullet j} = 0 \quad (4)$$

and

$$\ell_{j-1} = \frac{n}{\lambda_{j-1}} - \theta (k - j + 1) t_{\bullet j} = 0. \quad (5)$$

It is immediate from solving (5) for λ_{j-1} that we have

$$\lambda_{j-1} = \frac{n}{\theta(k-j+1)t_{\bullet j}}, \quad j = 2, \dots, k. \quad (6)$$

Since $\lambda_0 = 1$, we rewrite (4) as

$$\frac{nk}{\theta} - kt_{\bullet 1} - \sum_{j=2}^k (k-j+1)\lambda_{j-1}t_{\bullet j} = 0. \quad (7)$$

Substituting (6) into (7) gives

$$\frac{nk}{\theta} - kt_{\bullet 1} - \frac{n(k-1)}{\theta} = 0. \quad (8)$$

Solving (8) for θ , we obtain the MLE of θ , denoted by $\hat{\theta}$,

$$\hat{\theta} = \frac{n}{kt_{\bullet 1}} = \frac{n}{k \sum_{i=1}^n t_{i1}}. \quad (9)$$

The MLEs of λ_{j-1} , denoted by $\hat{\lambda}_{j-1}$, are also easily obtained by substituting (9) into (6)

$$\hat{\lambda}_{j-1} = \frac{kt_{\bullet 1}}{(k-j+1)t_{\bullet j}} = \frac{k \sum_{i=1}^n t_{i1}}{(k-j+1) \sum_{i=1}^n t_{ij}}, \quad j = 2, 3, \dots, k.$$

3 Singh-Sharma-Kumar load-sharing model

Consider k -component system connected in parallel. Following Singh et al. (2008), we assume the following:

- (i) A system is made up of k components whose lifetimes are independent and have exponential distributions with initial failure rate θ .

(ii) After the first component fails, the failure rates of the remaining $k - 1$ components change to $\lambda_1\theta$ where $\lambda_1 > 0$. After the next component failure, the failure rates of the surviving $k - 2$ components change to $\lambda_2\theta$. Then, after the next component failure, the the failure rates of the surviving $k - 3$ components change to $\lambda_3\theta$ and so on and so forth.

After the failure of a certain number of components, say, after the s -th component failure ($s \geq 2$), the failure rates of the $k - s$ remaining components change to $\lambda_s t\theta$ (linearly increasing failure rate). In a similar manner, after the $(s + 1)$ -th component failure, the failure rates of the $k - s - 1$ remaining components change to $\lambda_{s+1}t\theta$, and so on. Finally, after the last failure, the failure rate of the last component becomes $\lambda_{k-1}t\theta$.

(iii) There are n repeated measurements of independent systems. That is, we have a random sample of independent systems of size n .

Let T_{ij} denote the time spacing between the $(j - 1)$ -th failure and j -th failure for the i -th system where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, k$. As is given in Singh et al. (2008), the likelihood function for the i -th system is

$$L_i(\theta, \Lambda) = (k!) \theta^k \cdot \left[\prod_{j=1}^k \lambda_{j-1} \right] \cdot \left[\prod_{j=s+1}^k t_{ij} \right] \\ \times \exp \left[-\theta \left\{ \sum_{j=1}^s (k - j + 1) \lambda_{j-1} t_{ij} + \frac{1}{2} \sum_{j=s+1}^k (k - j + 1) \lambda_{j-1} t_{ij}^2 \right\} \right],$$

where $\lambda_0 = 1$ and $\Lambda = (\lambda_1, \dots, \lambda_{k-1})$. It is immediate that the likelihood function for a random sample of size n is given by

$$L(\theta, \Lambda) = (k!)^n \theta^{nk} \cdot \left[\prod_{j=1}^k \lambda_{j-1}^n \right] \cdot \left[\prod_{i=1}^n \prod_{j=s+1}^k t_{ij} \right] \\ \times \exp \left[-\theta \sum_{i=1}^n \left\{ \sum_{j=1}^s (k-j+1) \lambda_{j-1} t_{ij} + \frac{1}{2} \sum_{j=s+1}^k (k-j+1) \lambda_{j-1} t_{ij}^2 \right\} \right]. \quad (10)$$

Taking the logarithm of (10), differentiating with respect to $\theta, \lambda_1, \dots, \lambda_{k-1}$, denoting partial derivative of $\log L$ with respect to θ as $\ell_\theta = \partial \log L / \partial \theta$ and partial derivative of $\log L$ with respect to λ_{j-1} as $\ell_{j-1} = \partial \log L / \partial \lambda_{j-1}$, we obtain the log-likelihood estimating equations shown below:

$$\ell_\theta = \frac{nk}{\theta} - \sum_{i=1}^n \left\{ \sum_{j=1}^s (k-j+1) \lambda_{j-1} t_{ij} + \frac{1}{2} \sum_{j=s+1}^k (k-j+1) \lambda_{j-1} t_{ij}^2 \right\} = 0, \quad (11)$$

$$\ell_{j-1} = \frac{n}{\lambda_{j-1}} - \theta \sum_{i=1}^n (k-j+1) t_{ij} = 0, \quad j = 2, 3, \dots, s, \quad (12)$$

and

$$\ell_{j-1} = \frac{n}{\lambda_{j-1}} - \frac{\theta}{2} \sum_{i=1}^n (k-j+1) t_{ij}^2 = 0, \quad j = s+1, \dots, k. \quad (13)$$

Let us y_{ij} define as

$$y_{ij} = (k-j+1) t_{ij} I_{[j \leq s]} + \frac{1}{2} (k-j+1) t_{ij}^2 I_{[j > s]}, \quad (14)$$

where I_A is an indicator function whose value is one if A is satisfied and is zero if A is not satisfied. Then (11), (12) and (13) can be expressed as

$$\ell_\theta = \frac{nk}{\theta} - \sum_{i=1}^n \sum_{j=1}^k \lambda_{j-1} y_{ij} = 0, \quad (15)$$

and

$$\ell_{j-1} = \frac{n}{\lambda_{j-1}} - \theta \sum_{i=1}^n y_{ij} = 0, \quad j = 2, 3, \dots, k. \quad (16)$$

Notice that, by using (14), we have combined the two equations in (12) and (13) which resulted in the equation (16).

For convenience, let $y_{\bullet j} = \sum_{i=1}^n y_{ij}$. Using this and considering $\lambda_0 = 1$, we can rewrite (15) and (16) as follows:

$$\ell_\theta = \frac{nk}{\theta} - y_{\bullet 1} - \sum_{j=2}^k \lambda_{j-1} y_{\bullet j} = 0, \quad (17)$$

and

$$\ell_{j-1} = \frac{n}{\lambda_{j-1}} - \theta y_{\bullet j} = 0, \quad j = 2, 3, \dots, k. \quad (18)$$

Solving (18) for λ_{j-1} , we have

$$\lambda_{j-1} = \frac{n}{\theta y_{\bullet j}}, \quad j = 2, 3, \dots, k. \quad (19)$$

Substituting (19) into (17) gives

$$\frac{nk}{\theta} - y_{\bullet 1} - \frac{n(k-1)}{\theta} = 0. \quad (20)$$

Solving (20) for θ , we obtain the MLE of θ , denoted by $\hat{\theta}$,

$$\hat{\theta} = \frac{n}{y_{\bullet 1}} = \frac{n}{\sum_{i=1}^n y_{i1}}. \quad (21)$$

The MLEs of λ_{j-1} are also easily obtained by substituting (21) into (19)

$$\hat{\lambda}_{j-1} = \frac{y_{\bullet 1}}{y_{\bullet j}} = \frac{\sum_{i=1}^n y_{i1}}{\sum_{i=1}^n y_{ij}}, \quad j = 2, 3, \dots, k.$$

Notice that given the definition of y_{ij} , we can rewrite $y_{\bullet j}$ in the following manner:

$$y_{\bullet j} = \begin{cases} \sum_{i=1}^n (k-j+1)t_{ij} = (k-j+1) \sum_{i=1}^n t_{ij}, & j = 1, \dots, s \\ \frac{1}{2} \sum_{i=1}^n (k-j+1)t_{ij}^2 = \frac{1}{2}(k-j+1) \sum_{i=1}^n t_{ij}^2, & j = s+1, \dots, k \end{cases}.$$

Therefore we can write the closed form MLEs for θ and λ_{j-1} as

$$\hat{\theta} = \frac{n}{k \sum_{i=1}^n t_{i1}}$$

and

$$\hat{\lambda}_{j-1} = \begin{cases} \frac{k \sum_{i=1}^n t_{i1}}{(k-j+1) \sum_{i=1}^n t_{ij}}, & j = 1, \dots, s \\ \frac{k \sum_{i=1}^n t_{i1}}{\frac{1}{2}(k-j+1) \sum_{i=1}^n t_{ij}^2}, & j = s+1, \dots, k \end{cases} .$$

References

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