New Cumulative Damage Models for Failure Using Stochastic Processes as Initial Damage

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Abstract

Based on a generalized cumulative damage approach with a stochastic process describing "initial damage" for a material specimen, a broad class of statistical models for material strength is developed. Plausible choices of stochastic processes for the initial damage include Brownian motion, geometric Brownian motion, and the gamma process, and additive and multiplicative cumulative damage functions are considered. The resulting generalized statistical model gives an accelerated test form of the inverse Gaussian distribution, special cases of which include some existing models in addition to several new models. Model parameterizations and estimation by maximum likelihood from accelerated test data are discussed, and the applicability of the general model is illustrated for three sets of strength data.

Index Terms

Cumulative damage, Strength distribution, Inverse Gaussian distribution, Accelerated testing, Brownian motion, Gamma process.

ACRONYMS[1]

s-	implies: statistical(ly)
AIC	Akaike information criterion

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[1]The singular and plural of an acronym are always spelled the same.

Cdf	cumulative distribution function
pdf	probability density function
MLE	maximum-likelihood estimator
MSE	mean square error

NOTATION

$c(\cdot)$	damage accumulation function
$h(\cdot)$	damage model function
D	amount of damage; $D > 0$
D_n	damage to the specimen after n increments of stress
$F_S(\cdot)$	Cdf of S
$\hat{F}_{n_i}(\cdot)$	empirical Cdf of S at the gauge length L_i
$G_W(\cdot)$	Cdf of W
$H_c(\cdot)$	strength reduction model function
L	gauge length
N	number of stress increments until failure
n	a value of N; $n = 0, 1, 2, \ldots$
S	tensile strength; $S > 0$
s	a value of S
W	initial strength of material specimen
X_0	initial damage of material specimen
X_n	reduction in strength after n increments of stress
Y_u	cumulative initial damage at location u
$\ell(\boldsymbol{\theta})$	log-likelihood function
λ_L, μ_L	inverse Gaussian parameters
μ, σ	mean and standard deviation of D
ψ	theoretical strength
$\phi(\cdot)$	pdf of the standard *s*-normal distribution
$\Phi(\cdot)$	Cdf of the standard *s*-normal distribution
Ω_W	support of W

I. Introduction

STRENGTH properties of materials are required for engineering design of various structures in order for those structures to withstand anticipated stresses or loads. To gain knowledge of such properties, strengths of small specimens of the materials are often measured in laboratory settings, and appropriate mathematical or statistical models are required to predict strengths of different size specimens or structures than those tested. This scenario arises, for example, in the use of modern fibrous composite materials. Due to randomly occurring flaws in the material specimens themselves, perhaps from imperfections in the manufacturing process or handling, the tensile strength of a single specimen must be considered as a random variable whose probability distribution depends on the specimen size. Such a probability distribution is required in order to estimate the strength percentiles for use in the design of larger structures made from the material. Thus, statistical models that fit observed specimen data well are required to better describe and design larger structures.

It has been shown that the commonly used Weibull distributions often do not provide good fits to tensile strength data. For example, for carbon fiber or composite tensile strengths, see Durham & Padgett [1] and Wolstenholme [2]. Hence, other statistical models are needed that are based on more plausible physical or engineering assumptions, and that provide good fits to observed tensile strength data (see Onar & Padgett [3] for some further examples). For describing strength of single fibers or their composites and cycles to failure, in particular, some other probability distributions have been investigated to some extent. Recent papers include Durham & Padgett [1], Onar & Padgett [3], Taylor [4], Stoner, Edie & Durham [5], Padgett, Durham & Mason [6], Owen & Padgett [7], [8], and Padgett & Tomlinson [9].

In this paper, the results of Durham & Padgett [1], Padgett [10], Onar & Padgett [3], and Padgett & Tomlinson [9], among others, are unified into a single generalized model framework. First, a general class of statistical models based on cumulative damage is derived which is useful for accelerated testing situations. The approach assumes that "initial damage" exists in a material specimen which reduces its theoretical strength, and the initial damage can be modeled by a stochastic process that results in the distribution of the specimen's initial strength. Specific processes considered are Brownian motion, geometric Brownian motion, lognormal processes, and gamma processes. In each case, the process provides a plausible description of the existing flaws or other initial damage to the material specimen before it is subjected to strength testing. These processes together with the cumulative damage model functions result in strength distributions that turn out to be inverse Gaussian-type models incorporating the specimen size as an “acceleration variable.” New accelerated test models, as well as several existing models, are obtained as special cases of the general model.

The framework for the new generalized model is developed in Section II and special cases are presented in Section III. Parameterizations of the models and estimation procedures by maximum likelihood are discussed in Section IV. Applications of the models to well-known data sets on tensile strength of carbon “micro-composite” specimens and on carbon fiber tensile strengths are examined in Section V.

II. DERIVATION OF THE GENERAL STATISTICAL MODELS

Consider a material specimen of gauge length (or size) L with unknown theoretical tensile *strength* ψ, which is a fixed quantity. The specimen is placed under tensile load which is steadily increased until failure. We assume the following:

A1. The increasing load is discretized so the stress is incremented by small, discrete amounts until the specimen breaks, resulting in the observation of its breaking stress or strength.

A2. Each small increment of stress causes a non-negative *damage* amount D, which is a random variable with Cdf $F_D(\cdot)$, mean μ, and variance σ^2.

A3. The initial damage inherent in the specimen, before the load is applied, is in the form of the most severe “flaw” existing in the specimen and is quantified by a random variable, X_0, which results in a random reduction of ψ. The strength reduction is given by the strength reduction function $H_c(\cdot)$, which is strictly increasing and subject to the damage accumulation function $c(\cdot)$ described later. The difference of strength reduction functions, $H_c(\psi)-H_c(X_0)$, is almost surely greater than zero. Then, the *initial strength* of the specimen, W, is given by $W = H_c^{-1}\big(H_c(\psi) - H_c(X_0)\big)$. For example, $H_c(u) = u$ gives $W = \psi - X_0$ (linear reduction in initial strength), whereas $H_c(u) = \log u$ gives $W = \psi/X_0$ (geometric reduction in initial strength).

As the tensile load on the specimen is increased under the assumptions above, the cumulative damage after $n+1$ increments of stress [1], [11] is denoted by

$$X_{n+1} = X_n + D_n\, h(X_n), \tag{1}$$

where $D_j > 0$ for $j = 0, 1, 2, \ldots, n$ are the independent and identically distributed damages to the specimen at each increment and $h(u)$ is the damage model function. For example, $h(u) = 1$ gives an *additive* damage model, whereas $h(u) = u$ gives a *multiplicative* damage model.

We next consider the following generalized cumulative damage model:

$$c(X_{n+1}) = c(X_n) + D_n h(X_n), \tag{2}$$

where $c(\cdot)$ is a non-negative function. Since $D_n = [c(X_{n+1}) - c(X_n)]/h(X_n)$, the damage incurred to the specimen after n increments of stress is

$$\sum_{i=0}^{n-1} D_i = \sum_{i=0}^{n-1} \frac{c(X_{i+1}) - c(X_i)}{h(X_i)} \cong \int_{X_0}^{X_n} \frac{c'(x)}{h(x)}\, dx = H_c(X_n) - H_c(X_0)$$

for large n. Here $H_c(x) = \int \frac{c'(x)}{h(x)}\, dx$. Then, by the central limit theorem, $H_c(X_n) - H_c(X_0)$ has an approximate normal distribution with mean $n\mu$ and standard deviation $\sqrt{n}\sigma$.

Let N be the number of increments of tensile stress applied to a specimen of strength ψ until failure as before. Since $H_c(\cdot)$ is strictly increasing and $H_c(\psi) - H_c(X_0) > 0$ almost surely from A3, we have

$$\begin{aligned} N &= \sup_n \{n : X_1 \le \psi, \ldots, X_{n-1} \le \psi\} \\ &= \sup_n \{n : H_c(X_1) - H_c(X_0) \le H_c(\psi) - H_c(X_0), \ldots, \\ &\qquad\qquad H_c(X_{n-1}) - H_c(X_0) \le H_c(\psi) - H_c(X_0)\}, \end{aligned}$$

where $N = 1$ if the set is empty. Then, we have the following conditional probability,

$$P[N > n \mid H_c(\psi) - H_c(X_0) = w^*] = P[H_c(X_n) - H_c(X_0) \le w^*]$$

because $\{X_n\}$ is a nondecreasing sequence, *i.e.*, all $D_n \ge 0$ with probability 1 and where we let $w^* = H_c(w)$ for convenience. Then, the survival probability after n increments of stress is given by

$$P(N > n) = \int_{\Omega_W} F_n(w)\, dG_W(w), \tag{3}$$

where $F_n(w) = P[H_c(X_n) - H_c(X_0) \le H_c(w)]$ and $G_W(\cdot)$ is the distribution function of $W = H_c^{-1}\big(H_c(\psi) - H_c(X_0)\big)$. Therefore, for large n, we have

$$F_n(w) = P[H_c(X_n) - H_c(X_0) \le H_c(w)] \cong \Phi\Big(\frac{H_c(w) - n\mu}{\sqrt{n}\sigma}\Big), \tag{4}$$

where $\Phi(\cdot)$ denotes the standard normal Cdf.

For a specimen of gauge length L, let Y_u denote the cumulative initial damage at location u $(0 \le u \le L)$ along the length of the specimen. Let $M_L = \max\{Y_u : 0 \le u \le L\}$ denote the initial damage in terms of the severity of the inherent flaws over the specimen; that is, M_L is the random strength reduction of the specimen due to the most severe inherent flaw present before stress is applied to the specimen of gauge length L. Thus, the initial strength becomes $W = H_c^{-1}\big(H_c(\psi) - H_c(M_L)\big)$. It is noteworthy that $M_L = Y_L$ if $Y_t - Y_s$ is non-negative for $t > s$. Since $H_c(\cdot)$ is strictly increasing, the Cdf of $W = H_c^{-1}(H_c(\psi) - H_c(M_L))$ is given by

$$\begin{aligned} G_W(w) &= P\big[H_c^{-1}(H_c(\psi) - H_c(M_L)) \le w \mid w \in \Omega_W\big] \\ &= P\big[M_L \ge H_c^{-1}(H_c(\psi) - H_c(w)) \mid w \in \Omega_W\big] \\ &= \frac{1 - F_{M_L}\big(H_c^{-1}(H_c(\psi) - H_c(w))\big)}{P[0 < M_L < \psi]} \\ &= \frac{1 - F_{M_L}\big(H_c^{-1}(H_c(\psi) - H_c(w))\big)}{F_{M_L}(\psi) - F_{M_L}(0)}, \end{aligned}$$

where $\Omega_W = \{w : w = H_c^{-1}(H_c(\psi) - H_c(m)),\ 0 < m < \psi\}$. Thus, it follows that the pdf of W is

$$g_W(w) = \frac{H_c'(w)}{B} \cdot \frac{f_{M_L}\Big(H_c^{-1}\big(H_c(\psi) - H_c(w)\big)\Big)}{H_c'\Big(H_c^{-1}\big(H_c(\psi) - H_c(w)\big)\Big)}, \tag{5}$$

where $B = F_{M_L}(\psi) - F_{M_L}(0)$ and $H_c'(u) = dH_c(u)/du$. Therefore, the survival probability after n increments of stress is approximated by

$$P(N > n) \cong \int_{\Omega_W} \Phi\Big(\frac{H_c(w) - n\mu}{\sqrt{n}\sigma}\Big) g_W(w)\, dw = E\Big[\Phi\Big(\frac{H_c(W) - n\mu}{\sqrt{n}\sigma}\Big)\Big]. \tag{6}$$

Using a two-term Taylor's expansion of the above $\Phi(\cdot)$ function about $\big\{E\big(H_c(W)\big) - n\mu\big\}/(\sqrt{n}\sigma)$ and taking the expectation with respect to W, we have

$$E\Big[\Phi\Big(\frac{H_c(W) - n\mu}{\sqrt{n}\sigma}\Big)\Big] \cong \Phi\Big(\frac{E\big(H_c(W)\big)}{\sqrt{n}\sigma} - \frac{\sqrt{n}\mu}{\sigma}\Big),$$

where the expectation of $H_c(W)$, denoted by $\Lambda(\boldsymbol{\theta}; L)$, is given by

$$\Lambda(\boldsymbol{\theta}; L) = E\big(H_c(W)\big) = \int_{\Omega_W} H_c(w) \cdot g_W(w)\, dw. \tag{7}$$

Note that this expectation is an acceleration function with an unknown parameter (possibly a vector) and depends on the known gauge length L (*i.e.*, the acceleration variable) from (5).

Finally, letting S be a continuous version of N and using the symmetry of $\Phi(\cdot)$ gives the failure distribution of the specimen as

$$F_S(s) = P(S \le s) \cong \Phi\Big(\frac{\sqrt{s}\mu}{\sigma} - \frac{\Lambda(\boldsymbol{\theta}; L)}{\sqrt{s}\sigma}\Big).$$

Reparameterizing as $\mu_L = \Lambda(\boldsymbol{\theta}; L)/\mu$ and $\lambda_L = \big[\Lambda(\boldsymbol{\theta}; L)/\sigma\big]^2$, we have

$$F_S(s) = P(S \le s) = \Phi\Big[\sqrt{\frac{\lambda_L}{s}}\Big(\frac{s}{\mu_L} - 1\Big)\Big] = \Phi\Big[\sqrt{\frac{\lambda_L}{\mu_L}}\Big(\sqrt{\frac{s}{\mu_L}} - \sqrt{\frac{\mu_L}{s}}\,\Big)\Big]. \tag{8}$$

Note that the equation (8) is a Birnbaum-Saunders [12] type distribution that incorporates the gauge length L. The difference between the inverse Gaussian and Birnbaum-Saunders distributions is negligible when $\sqrt{\lambda_L} \gg \mu_L$ [13]. So Birnbaum-Saunders models can be approximated by the first term of an inverse Gaussian when $\xi = \sqrt{\lambda_L}/\mu_L \gg 1$. The pdf for (8) is approximated by an inverse Gaussian with mean parameter μ_L and scale λ_L, which is given by

$$f_S(s) \cong \sqrt{\frac{\lambda_L}{2\pi s^3}} \exp\Big[-\frac{\lambda_L(s-\mu_L)^2}{2\mu_L^2 s}\Big], \quad s > 0.$$

We denote the tensile strength S then as

$$S \mathrel{\dot\sim} \mathrm{IG}\big(\lambda_L, \mu_L\big),$$

where $\mu_L = \Lambda(\boldsymbol{\theta}; L)/\mu$ and $\lambda_L = \big[\Lambda(\boldsymbol{\theta}; L)/\sigma\big]^2$. Note that the reciprocal of the inverse Gaussian random variable is also inverse Gaussian. That is, if the random variable S is inverse Gaussian with the parameters λ_L and μ_L, then we have $S^* = 1/S \sim \mathrm{IG}\big(\lambda_L^*, \mu_L^*\big)$ with $\lambda_L^* = \lambda_L/\mu_L^2$ and $\mu_L^* = 1/\mu_L$.

By selecting various forms of the functions $c(\cdot)$ and $h(\cdot)$, several new models can be obtained. This is described in the following section. It is also worth mentioning that other models can be obtained by using any reasonable distribution rather than the Cdf of the initial strength subject to $W = H_c^{-1}\big(H_c(\psi) - H_c(X_0)\big)$. Using this relaxed Cdf, we can have other models such as "Gauss-Weibull additive model" [1] and "Gauss-Gauss multiplicative model" [10]. A sequel paper will deal with other models using a relaxed Cdf of W.

III. Special Cases

Here we use specific choices of the functions $c(\cdot)$ and $h(\cdot)$ to obtain additive or multiplicative damage models for the Cdf $F_S(s)$. Also, for the initial damage, the initial strength distributions are obtained based on a variety of stochastic processes including Brownian motion, geometric Brownian motion, lognormal and gamma processes. We consider two damage models: *additive* and *multiplicative* models. These combinations yield several different acceleration functions from (7).

A. Additive Damage Model (Linear Reduction Model)

We first consider the additive damage model (or linear strength reduction model). Using the strength reduction function $H_c(u) = u$ and the damage cumulation function $c(u) = u$, we have the following cumulative damage model

$$X_{n+1} = X_n + D_n.$$

This is an additive model. We derive the initial strength distributions using Brownian motion, geometric Brownian motion, lognormal and gamma processes as initial damage.

1) Brownian Motion Process: We assume that the stochastic process $\{Y_u : 0 \le u \le L\}$ is a Brownian motion with positive drift coefficient α and diffusion β^2. Then, the random strength reduction of the specimen is $M_L = \max\{Y_u : 0 \le u \le L\}$. To obtain the pdf of $W = \psi - M_L$, we need to find the pdf of M_L. Shepp [14] gives the following joint pdf of $Z = M_L$ and $X = Y_L$ of a Brownian motion process:

$$g(z,x) = \frac{2(2z-x)}{\sqrt{2\pi\beta^6 L^3}} \exp\Big(\frac{\alpha x}{\beta^2} - \frac{\alpha^2 L}{2\beta^2}\Big) \exp\Big(-\frac{(2z-x)^2}{2\beta^2 L}\Big), \tag{9}$$

where $z > 0$ and $x < z$. By integrating $g(z,x)$ with respect to x over $(-\infty, z)$, we obtain the pdf of M_L

$$f_{M_L}(z) = \frac{2}{\beta\sqrt{L}}\phi\Big(\frac{z-\alpha L}{\beta\sqrt{L}}\Big) - \frac{2\alpha}{\beta^2}\Phi\Big(-\frac{z+\alpha L}{\beta\sqrt{L}}\Big)\exp\Big(\frac{2\alpha z}{\beta^2}\Big). \tag{10}$$

The above pdf can be approximated by the first term on the right hand side when $\beta \gg \alpha$, so that we have

$$f_{M_L}(z) \cong \frac{2}{\beta\sqrt{L}}\phi\Big(\frac{z-\alpha L}{\beta\sqrt{L}}\Big). \tag{11}$$

Substituting the above into (5), we have the pdf of W

$$g_W(w) = \frac{2}{B\beta\sqrt{L}}\phi\Big(\frac{w-(\psi-\alpha L)}{\beta\sqrt{L}}\Big), \quad 0 < w < \psi$$

where $B = 2\Phi\big(\frac{\psi-\alpha L}{\beta\sqrt{L}}\big) - 2\Phi\big(\frac{-\alpha L}{\beta\sqrt{L}}\big)$. Using the following integral identity,

$$\int v\cdot\phi\Big(\frac{v-a}{b}\Big)\,dv = -b^2\phi\Big(\frac{v-a}{b}\Big) + a\,b\,\Phi\Big(\frac{v-a}{b}\Big),$$

we have

$$E\big(H_c(W)\big) = \psi - \alpha L - \frac{2\beta\sqrt{L}}{B}\Big\{\phi\Big(\frac{\alpha L}{\beta\sqrt{L}}\Big) - \phi\Big(\frac{\psi-\alpha L}{\beta\sqrt{L}}\Big)\Big\}.$$

For large ψ and $\beta \gg \alpha$, we have

$$E\big(H_c(W)\big) \cong \psi - \theta\sqrt{L} - \alpha L,$$

where $\theta = \sqrt{2/\pi}\cdot\beta$. This model includes the "Gauss-Gauss additive model" [1] as a special case when $\alpha = 0$ and $\beta = 1$.

2) Geometric Brownian Motion Process: Next, we assume that the stochastic process $\{Y_u : 0 \leq u \leq L\}$ is a geometric Brownian motion (see Ross [15]), that is, $\{\log(Y_u) : 0 \leq u \leq L\}$ is a stationary Brownian motion with drift coefficient α and diffusion β^2. Then, the random strength reduction of the specimen is $M_L = \max\{Y_u : 0 \leq u \leq L\}$. To obtain the pdf of $W = \psi - M_L$, we need to find the pdf of M_L. Let $Z^* = \log M_L$, then the joint pdf of Z^* and $X = Y_L$ can be obtained by using Shepp [14] as before. By integrating the joint pdf with respect to X and changing the variable $Z = \exp(Z^*)$, we have the following pdf of W when $\beta \gg \alpha$

$$g_W(w) = \frac{2}{B\beta\sqrt{L}(\psi - w)}\phi\Big(\frac{\log(\psi - w) - \alpha L}{\beta\sqrt{L}}\Big), \quad 0 < w < \psi$$

where $B = 2\Phi\big(\frac{\log\psi - \alpha L}{\beta\sqrt{L}}\big)$. Thus, we have

$$E\big(H_c(W)\big) = \frac{2}{B\beta\sqrt{L}}\int_0^{\psi} \frac{w}{\psi - w}\cdot\phi\Big(\frac{\log(\psi - w) - \alpha L}{\beta\sqrt{L}}\Big)\,dw.$$

Changing the integrating variable to $u = \log(\psi - w)$ gives

$$\begin{aligned}
E\big(H_c(W)\big) &= \psi - \frac{2}{B\beta\sqrt{L}}\int_{-\infty}^{\log\psi} e^u \phi\Big(\frac{u - \alpha L}{\beta\sqrt{L}}\Big)\,du \\
&= \psi - \frac{2}{B\beta\sqrt{L}}\exp\Big[(\alpha + \frac{1}{2}\beta^2)L\Big]\int_{-\infty}^{\log\psi} \phi\Big(\frac{u - (\alpha + \beta^2)L}{\beta\sqrt{L}}\Big)\,du \\
&= \psi - \exp\Big[(\alpha + \frac{1}{2}\beta^2)L\Big]\frac{\Phi\Big(\frac{\log\psi - (\alpha + \beta^2)L}{\beta\sqrt{L}}\Big)}{\Phi\big(\frac{\log\psi - \alpha L}{\beta\sqrt{L}}\big)} \\
&\cong \psi - \exp\Big[(\alpha + \frac{1}{2}\beta^2)L\Big]\frac{\Phi\Big(\frac{\log\psi}{\beta\sqrt{L}} - \beta\sqrt{L}\Big)}{\Phi\big(\frac{\log\psi}{\beta\sqrt{L}}\big)}
\end{aligned}$$

If we assume that ψ is large, then we have

$$E\big(H_c(W)\big) \cong \psi - e^{\theta L},$$

where $\theta = \alpha + \beta^2/2$.

3) Lognormal Process: Next, we assume that the process $\{Y_u : 0 \leq u \leq L\}$ is a lognormal process, that is, $\log(Y_{s+t} - Y_s)$ is normally distributed with the parameters αt and $\beta^2 t$ for all $s, t > 0$. Notice that for a geometric Brownian motion process, $\log Y_{s+t} - \log Y_s$ is normally distributed. Thus, a lognormal process Y_u is strictly increasing while a geometric Brownian motion is not. Since Y_u is strictly increasing in u, the random strength reduction of the specimen is $M_L = Y_L$. Thus, it follows from (5) that the pdf of $W = \psi - Y_L$ is

$$g_W(w) = \frac{1}{B\beta\sqrt{L}(\psi - w)}\phi\Big(\frac{\log(\psi - w) - \alpha L}{\beta\sqrt{L}}\Big), \quad 0 < w < \psi$$

where $B = \Phi\big(\frac{\log\psi - \alpha L}{\beta\sqrt{L}}\big)$. The above pdf is a form for the previous geometric Brownian motion process case. Hence, it gives the same form of $E\big(H_c(W)\big)$ as the geometric Brownian motion case. Notice that the above pdf is obtained without any approximation, but the pdf of W for the geometric Brownian motion case is obtained based on an approximate $f_{M_L}(\cdot)$ assuming $\beta \gg \alpha$.

4) Gamma Process: For another model, we next assume that the stochastic process $\{Y_u : 0 \leq u \leq L\}$ is a gamma process with the shape parameter αu and scale β. Since Y_u is strictly increasing in u, the random strength reduction of the specimen is $M_L = Y_L$. Thus, it follows from (5) that the pdf of $W = \psi - Y_L$ is

$$g_W(w) = \frac{1}{B_{\alpha L}\Gamma(\alpha L)\beta^{\alpha L}}(\psi - w)^{\alpha L - 1}\exp\Big(-\frac{\psi - w}{\beta}\Big), \quad 0 < w < \psi$$

where $B_{\alpha L} = \int_0^\psi f_{\alpha L}(y)\,dy$ and $f_{\alpha L}(\cdot)$ is the pdf of gamma with parameters αL and β. It follows that

$$E\big(H_c(W)\big) = \int_0^\psi w \cdot g_W(w)\,dw = \psi - \int_0^\psi (\psi - w)g_W(w)\,dw.$$

Changing the integrating variable to $u = \psi - w$ gives

$$E\big(H_c(W)\big) = \psi - \beta\frac{\Gamma(\alpha L + 1)}{\Gamma(\alpha L)}\frac{B_{\alpha L}}{B_{\alpha L+1}}.$$

For large ψ, we have $B_{\alpha L} \cong B_{\alpha L+1}$ so that

$$E\big(H_c(W)\big) \cong \psi - \theta L,$$

where $\theta = \alpha\beta$. This result is the same as Padgett & Tomlinson [9].

B. Multiplicative Damage Model (Geometric Reduction Model)

We next consider the multiplicative damage model (or geometric strength reduction model). Using the strength reduction function $H_c(u) = \log u$ and the damage cumulation function $c(u) = u$, we have the following cumulative damage model

$$X_{n+1} = X_n + D_n X_n.$$

This is a multiplicative damage model. We derive the initial strength distributions again using Brownian motion, geometric Brownian motion, lognormal and gamma processes as initial damage.

1) Brownian Motion Process: We again assume that the stochastic process $\{Y_u : 0 \leq u \leq L\}$ is a Brownian motion with positive drift coefficient α and diffusion β^2. The random strength reduction of the specimen is $M_L = \max\{Y_u : 0 \leq u \leq L\}$. Then, the pdf of M_L is given by (11) when $\beta \gg \alpha$, and using the equation (5), we have

$$g_W(w) = \frac{\psi}{Bw^2} \cdot f_{M_L}\Big(\frac{\psi}{w}\Big) = \frac{\psi}{B\beta\sqrt{L}w^2}\phi\Big(\frac{\psi/w - \alpha L}{\beta\sqrt{L}}\Big), \quad w > 1$$

where $B = \Phi\big(\frac{\psi-\alpha L}{\beta\sqrt{L}}\big) - \Phi\big(\frac{-\alpha L}{\beta\sqrt{L}}\big)$. Thus, we have

$$E\big(H_c(W)\big) = \int_1^\infty \log w \cdot g_W(w)\, dw.$$

Changing the integrating variable to $v = \psi - \psi/w$ gives

$$E\big(H_c(W)\big) = \int_0^\psi \Big(\log\psi - \log(\psi - v)\Big) \cdot \frac{1}{B\beta\sqrt{L}}\phi\Big(\frac{\psi - v - \alpha L}{\beta\sqrt{L}}\Big)\, dv.$$

Considering a two-term Taylor series approximation,

$$\log(1+x) \cong x - \frac{1}{2}x^2 \quad (-1 < x \leq 1),$$

and substituting $x = -v/\psi$ into the above, we have

$$\log(\psi - v) \cong \log\psi - \frac{v}{\psi} - \frac{v^2}{2\psi^2}.$$

Thus, we have

$$E\big(H_c(W)\big) \cong \int_0^\psi \Big(\frac{v}{\psi} + \frac{v^2}{2\psi^2}\Big) \cdot \frac{1}{B\beta\sqrt{L}}\phi\Big(\frac{v - (\psi - \alpha L)}{\beta\sqrt{L}}\Big)\, dv.$$

Using the following integral identities,

$$\int v \cdot \phi\Big(\frac{v-a}{b}\Big)\, dv = -b^2\phi\Big(\frac{v-a}{b}\Big) + a\, b\, \Phi\Big(\frac{v-a}{b}\Big),$$
$$\int v^2 \cdot \phi\Big(\frac{v-a}{b}\Big)\, dv = -b^2(a+v)\phi\Big(\frac{v-a}{b}\Big) + b(a^2+b^2)\Phi\Big(\frac{v-a}{b}\Big),$$

we have

$$E\big(H_c(W)\big) \cong \frac{a}{\psi} + \frac{a^2+b^2}{2\psi^2} + \frac{b}{B\psi}(1 + \frac{a}{2\psi})\phi(\frac{-a}{b}) - \frac{b}{B\psi}(1 + \frac{a+\psi}{2\psi})\phi(\frac{\psi - a}{b}),$$

where $a = \psi - \alpha L$ and $b = \beta\sqrt{L}$. Assuming that ψ is large, we have

$$\begin{aligned} E\big(H_c(W)\big) &\cong \frac{\psi - \alpha L}{\psi} + \frac{(\psi - \alpha L)^2 + \beta^2 L}{2\psi^2} - \frac{\beta\sqrt{L}}{B\psi}\Big(1 + \frac{2\psi - \alpha L}{2\psi}\Big)\phi\Big(\frac{\alpha L}{\beta\sqrt{L}}\Big) \\ &= \frac{3}{2} - \frac{2\alpha L}{\psi} + \frac{\alpha^2L^2 + \beta^2 L}{2\psi^2} - \frac{\beta\sqrt{L}}{B\psi}\Big(1 + \frac{2\psi - \alpha L}{2\psi}\Big)\phi\Big(\frac{\alpha L}{\beta\sqrt{L}}\Big). \end{aligned}$$

If $\alpha \ll \beta$, then $B \cong \frac{1}{2}$ and thus we have

$$E\big(H_c(W)\big) \cong \frac{3}{2} - \sqrt{\frac{8}{\pi}}\frac{\beta}{\psi}\sqrt{L} + \Big(\frac{1}{2}\frac{\beta^2}{\psi^2} - \frac{2\alpha}{\psi}\Big)L = \frac{3}{2} - \theta\sqrt{L} + \eta L,$$

where $\theta = \sqrt{\frac{8}{\pi}}\frac{\beta}{\psi}$ and $\eta = \frac{1}{2}\frac{\beta^2}{\psi^2} - \frac{2\alpha}{\psi}$.

2) Geometric Brownian Motion Process: Let the process $\{Y_u : 0 \leq u \leq L\}$ be a geometric Brownian motion with drift coefficient α and diffusion β^2. So the random strength reduction of the specimen is $M_L = \max\{Y_u : 0 \leq u \leq L\}$. As before, the joint pdf of $Z^* = \log M_L$ and $X = Y_L$ can be obtained by using Shepp [14], and from (5), we have the pdf of $W = \psi/M_L$ when $\beta \gg \alpha$,

$$g_W(w) = \frac{1}{B\beta\sqrt{L}w}\phi\Big(\frac{\log w - \log\psi + \alpha L}{\beta\sqrt{L}}\Big), \quad w > 1$$

where $B = \Phi\Big(\frac{\log\psi - \alpha L}{\beta\sqrt{L}}\Big)$. Thus, we have

$$E\big(H_c(W)\big) = \int_1^\infty \log w \cdot g_W(w)\, dw.$$

Changing the integrating variable to $v = \log w$ gives

$$E\big(H_c(W)\big) = \frac{1}{B\beta\sqrt{L}}\int_1^\infty v \cdot \phi\Big(\frac{v - \log\psi + \alpha L}{\beta\sqrt{L}}\Big)\, dv.$$

Using the following integral identity,

$$\int v \cdot \phi\Big(\frac{v-a}{b}\Big)\, dv = -b^2\phi\Big(\frac{v-a}{b}\Big) + a\, b\, \Phi\Big(\frac{v-a}{b}\Big),$$

we have

$$E\big(H_c(W)\big) = \frac{1}{B}\left[\beta\sqrt{L}\phi\Big\{\frac{\log\psi - \alpha L - 1}{\beta\sqrt{L}}\Big\} + (\log\psi - \alpha L)\cdot\Phi\Big\{\frac{\log\psi - \alpha L - 1}{\beta\sqrt{L}}\Big\}\right].$$

For large ψ, then

$$E\big(H_c(W)\big) \cong \log\psi - \alpha L.$$

3) Lognormal Process: As with the additive damage model in Section III-A, the pdf of W here has the same form of the geometric Brownian motion process case. Hence, it gives the same form of $E\big(H_c(W)\big)$.

4) Gamma Process: Again, letting $\{Y_u : 0 \le u \le L\}$ be a gamma process with the shape parameter αu and scale β, the random strength reduction of the specimen is $M_L = Y_L$. Thus, it follows from (5) that the pdf of $W = \psi/Y_L$ is

$$\begin{aligned} g_W(w) &= \frac{\psi}{B_{\alpha L} w^2} f_{\alpha L}\Big(\frac{\psi}{w}\Big) \\ &= \frac{\psi}{B_{\alpha L} w^2} \frac{1}{\Gamma(\alpha L)\beta^{\alpha L}} \Big(\frac{\psi}{w}\Big)^{\alpha L - 1} \exp\Big(-\frac{\psi}{\beta w}\Big) \\ &= \frac{(\psi/\beta)^{\alpha L}\big(1/w\big)^{\alpha L+1}}{\Gamma(\alpha L)} \exp\Big(-\frac{\psi}{\beta w}\Big), \quad w > 1 \end{aligned}$$

where $B_{\alpha L} = \int_0^{\psi} f_{\alpha L}(y)\,dy$. Notice that the above pdf is a form of the truncated inverted gamma distribution with scale parameter ψ/β and shape αL. Thus, we have the expectation of $H_c(W)$

$$\begin{aligned} E\big(H_c(W)\big) &= \int_1^{\infty} \log w \cdot g_W(w)\,dw \\ &= \frac{\psi}{B_{\alpha L}\Gamma(\alpha L)\beta^{\alpha L}} \int_1^{\infty} \log w \cdot \Big(\frac{\psi}{w}\Big)^{\alpha L-1} \exp\Big(-\frac{\psi}{\beta w}\Big)\frac{1}{w^2}\,dw. \end{aligned}$$

Changing the integrating variable to $v = \psi - \psi/w$ gives

$$E\big(H_c(W)\big) = \frac{1}{B_{\alpha L}\Gamma(\alpha L)\beta^{\alpha L}} \int_0^{\psi} \Big(\log\psi - \log(\psi - v)\Big)(\psi - v)^{\alpha L-1} \exp\Big(-\frac{1}{\beta}(\psi - v)\Big)\,dv.$$

Considering a two-term Taylor series approximation,

$$\log(1+x) \cong x - \frac{1}{2}x^2 \quad (-1 < x \le 1),$$

and substituting $x = -v/\psi$ into the above, we obtain

$$\log(\psi - v) \cong \log\psi - \frac{v}{\psi} - \frac{v^2}{2\psi^2}.$$

Thus,

$$\begin{aligned} E\big(H_c(W)\big) &\cong \frac{1}{B_{\alpha L}\Gamma(\alpha L)\beta^{\alpha L}} \int_0^{\psi} \Big(\frac{v}{\psi} + \frac{v^2}{2\psi^2}\Big)(\psi - v)^{\alpha L-1} \exp\Big(-\frac{1}{\beta}(\psi - v)\Big)\,dv \\ &= \frac{b^{\alpha L}}{B_{\alpha L}\Gamma(\alpha L)} \int_0^1 \Big(u + \frac{1}{2}u^2\Big)(1-u)^{\alpha L-1} \exp\big(-b(1-u)\big)\,du \\ &= \frac{1}{B_{\alpha L}\Gamma(\alpha L)} \Big[\frac{3}{2}\Gamma\big(\alpha L, b(1-u)\big) - \frac{2}{b}\Gamma\big(\alpha L+1, b(1-u)\big) + \frac{1}{2b^2}\Gamma\big(\alpha L+2, b(1-u)\big)\Big]_{u=0}^{u=1}, \end{aligned}$$

where $b = \psi/\beta$ and the incomplete gamma function $\Gamma(y,z)$ is defined by the integral $\Gamma(y,z) = \int_z^{\infty} t^{y-1}e^{-t}\,dt$. Using $\Gamma(y,0) = \Gamma(y)$, it follows that

$$E\big(H_c(W)\big) \cong \frac{1}{B_{\alpha L}}\Big[\frac{3}{2}\Big\{1 - \frac{\Gamma(\alpha L, b)}{\Gamma(\alpha L)}\Big\} - \frac{2}{b}\Big\{\alpha L - \frac{\Gamma(\alpha L+1, b)}{\Gamma(\alpha L)}\Big\} + \frac{1}{2b^2}\Big\{\alpha L(\alpha L+1) - \frac{\Gamma(\alpha L+2, b)}{\Gamma(\alpha L)}\Big\}\Big].$$

For large ψ (*i.e.*, large b), we have $B_{\alpha L} \cong 1$, $\Gamma(\alpha L, b)/\Gamma(\alpha L) \cong 0$, $\Gamma(\alpha L+1, b)/\Gamma(\alpha L) \cong 0$, and $\Gamma(\alpha L+2, b)/\Gamma(\alpha L) \cong 0$. Hence, we have

$$E\big(H_c(W)\big) \cong \frac{3}{2} - \Big(\frac{2\alpha\beta}{\psi} - \frac{\alpha\beta^2}{2\psi^2}\Big)L + \frac{\alpha^2\beta^2}{2\psi^2}L^2 \cong \frac{3}{2} - \theta L + \eta L^2.$$

IV. Estimation of Parameters

In this section, we discuss estimation of the parameters of the proposed models. In Table I, we present the six acceleration functions obtained in Section III.

TABLE I

Acceleration Functions $\Lambda(\boldsymbol{\theta}; L)$

Initial damage	Linear reduction	Geometric reduction
Brownian motion	M_1: $\psi - \theta\sqrt{L} - \alpha L$	M_4: $\frac{3}{2} - \theta\sqrt{L} + \eta L$
Geometric Brownian motion	M_2: $\psi - e^{\theta L}$	M_5: $\log\psi - \theta L$
Gamma	M_3: $\psi - \theta L$	M_6: $\frac{3}{2} - \theta L + \eta L^2$

First, consider the parameters of the six models above. For the model M_1, the tensile strength S has $S \sim \mathrm{IG}(\mu_L, \lambda_L)$, where

$$\mu_L = \frac{\psi - \theta\sqrt{L} - \alpha L}{\mu} \quad \text{and} \quad \sqrt{\lambda_L} = \frac{\psi - \theta\sqrt{L} - \alpha L}{\sigma}.$$

With the above setup, parameter estimation is indeterminate. Reparameterizing $\theta_1 = \psi/\sigma$, $\theta_2 = \theta/\sigma$, $\theta_3 = \alpha/\sigma$ and $\xi = \mu/\sigma$, we have the following

$$\mu_L = (\theta_1 - \theta_2\sqrt{L} - \theta_3 L)/\xi \quad \text{and} \quad \sqrt{\lambda_L} = \theta_1 - \theta_2\sqrt{L} - \theta_3 L.$$

Therefore, it is easily seen that the models M_1 & M_4 are equivalent. Similarly, the models M_3 and M_5 are equivalent and reparameterized as

$$\mu_L = (\theta_1 - \theta_2 L)/\xi \quad \text{and} \quad \sqrt{\lambda_L} = \theta_1 - \theta_2 L.$$

For convenience, we reparameterize the model M_2 to

$$\mu_L = (\theta_1 - e^{\theta_2 L + \theta_3})/\xi \quad \text{and} \quad \sqrt{\lambda_L} = \theta_1 - e^{\theta_2 L + \theta_3},$$

and the model M_6 needs to be changed to

$$\mu_L = (\theta_1 - \theta_2 L - \theta_3 L^2)/\xi \quad \text{and} \quad \sqrt{\lambda_L} = \theta_1 - \theta_2 L - \theta_3 L^2.$$

Hence, we have reduced the six models to the following three models, which are summarized as

$$\begin{aligned} M_A: &\quad \lambda_L = (\theta_1 - \theta_2\sqrt{L} - \theta_3 L)^2 \\ M_B: &\quad \lambda_L = (\theta_1 - \theta_2 L - \theta_3 L^2)^2 \\ M_C: &\quad \lambda_L = (\theta_1 - e^{\theta_2 L + \theta_3})^2, \end{aligned}$$

where $\mu_L = \sqrt{\lambda_L}/\xi$ for all three models above. Note that the model M_A includes the "Gauss-Gauss additive model" [1] as a special case when $\theta_3 = 0$ and the "Gauss-Gauss multiplicative model" [10] as a special case with the constraint $\theta_1 = \{\log(\theta_2/\theta_3) - \log\sqrt{8\pi}\}\theta_2^2/(4\theta_3)$; and the model M_B includes "Gauss-gamma additive model" [9] as a special case when $\theta_3 = 0$.

For parameter estimation, suppose that subjects are tested at the k different gauge lengths, $L_1, L_2, \ldots, L_k$. At each gauge length L_i, tests are performed for n_i specimens, resulting in observed breaking strengths s_{ij}, $j = 1, 2, \ldots, n_i$ for $i = 1, 2, \ldots, k$. For estimation over *all* of the k gauge lengths, the MLE of the unknown model parameters in a given model can be obtained by maximizing the log-likelihood function

$$\ell(\boldsymbol{\theta}) = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \log f_S(s_{ij}; \mu_{L_i}, \lambda_{L_i}).$$

The log-likelihood function for the inverse Gaussian distribution is

$$\ell(\boldsymbol{\theta}) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\Big(\log\lambda_{L_i} - s_{ij}\xi^2 + 2\xi\sqrt{\lambda_{L_i}} - \frac{1}{s_{ij}}\lambda_{L_i} - C_{ij}\Big),$$

where $C_{ij} = \log(2\pi) + 3\log s_{ij}$. The MLE of ξ and θ_r for $r = 1, 2, 3$ are obtained by solving the following equations simultaneously

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial\xi} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\Big(-s_{ij}\xi + \sqrt{\lambda_{L_i}}\Big) = 0 \tag{12}$$

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial\theta_r} = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{n_i}\Big(\frac{1}{\lambda_{L_i}} + \frac{\xi}{\sqrt{\lambda_{L_i}}} - \frac{1}{s_{ij}}\Big)\frac{\partial\lambda_{L_i}}{\partial\theta_r} = 0. \tag{13}$$

The equation (12) can be easily solved for ξ so that the MLE of ξ is given by substituting the MLE of θ_r, $r = 1, 2, 3$ into

$$\hat{\xi} = \frac{\sum_{i=1}^{k} n_i \sqrt{\hat{\lambda}_{L_i}}}{\sum_{i=1}^{k} \sum_{j=1}^{n_i} s_{ij}}.$$

The MLE of θ_r, $r = 1, 2, 3$, depend on the actual form of the acceleration model (or λ_{L_i}), and they can be obtained by solving the simultaneous nonlinear likelihood estimation equations in (13). In most cases, however, these solutions must be obtained numerically. For numerical root-finding algorithms, the initial values for the parameters are necessary. An easy "least squares method" for obtaining the initial values is briefly described [1]. For each $i = 1, \ldots, k$, denote the ordered observations s_{ij} by $s_{i(j)}$. Then using an empirical estimate of $F_S(s_{i(j)})$ in (8) denoted by $\hat{F}_{n_i}(s_{i(j)})$, we set

$$\hat{F}_{n_i}(s_{i(j)}) = \Phi\Big(\xi\sqrt{s_{i(j)}} - \sqrt{\lambda_{L_i}}\frac{1}{\sqrt{s_{i(j)}}}\Big),$$

which yields the following "linearized" equation

$$\sqrt{s_{i(j)}} \cdot \Phi^{-1}\Big(\hat{F}_{n_i}(s_{i(j)})\Big) = \xi s_{i(j)} - \sqrt{\lambda_{L_i}}.$$

Several versions of these empirical estimates $\hat{F}_{n_i}(s_{i(j)})$ have been suggested in the statistics literature, but the most popular one is $(j - 1/2)/n$ (also known as the *median rank method*) for $n \geq 11$ and $(j - 3/8)/(n + 1/4)$ for $n \leq 10$, due to Blom [16] and Wilk & Gnanadesikan [17].

For example, using the model M_A, we have

$$\sqrt{s_{i(j)}} \cdot \Phi^{-1}\Big(\frac{j - 1/2}{n_i}\Big) = \xi s_{i(j)} - \big(\theta_1 - \theta_2\sqrt{L_i} - \theta_3 L_i\big), \tag{14}$$

where for convenience, we assume $n \geq 11$. Thus, (14) can be used to find least squares estimates of θ_1, θ_2 and θ_3 from any statistical software, providing some initial values for the MLE equations in (13). Similarly, we can apply this method to the models M_B and M_C. For the model M_C, the ordinary linear least squares method is not applicable to this model for obtaining the initial values. However, using a non-linear regression, we can find it. Most statistical software packages offer non-linear least squares estimates such as the `nls` $(\cdot)$ function in the `R` [18] language.

The proposed models will be applied in the following section for real-data examples. A model selection procedure based on AIC [19], [20], [21] is also provided.

V. Examples Of Application Of The Models

In this section, we illustrate three real-data examples. The data analysis is performed using the R language, which is an open source software for statistical computing and graphics originally developed by Ihaka and Gentleman [18]. This can be obtained at no cost from http://www.r-project.org/ .

To compare the fits of the proposed models, the *overall* MSE from the fitted model to the empirical distribution, over all gauge lengths of L_i, $i = 1, 2, \ldots, k$, were used. Letting $\hat{F}_{n_i}(s_{ij})$ denote the empirical Cdf and $F_S(s_{ij}; \hat{\boldsymbol{\theta}})$ denote the fitted Cdf using the MLE of $\boldsymbol{\theta}$, the MSE for the fitted model is calculated as

$$\mathrm{MSE}\big(F_S(\cdot; \hat{\boldsymbol{\theta}})\big) = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{n_i}\sum_{j=1}^{n_i}\big\{F_S(s_{ij}; \hat{\boldsymbol{\theta}}) - \hat{F}_{n_i}(s_{ij})\big\}^2.$$

For a model selection procedure, we also report the AIC defined by

$$\mathrm{AIC} = -2 \times (\text{maximum log-likelihood}) + 2p,$$

where p is the number of independent model parameters. This AIC is frequently used in engineering and statistics literature to give a guideline for a model selection when there are several potential models by selecting the one with the smallest AIC among them [21].

Padgett, Durham & Mason [6] investigated the power-law Weibull model as an *overall* better fitting size-effect model than the ordinary Weibull model. The power-law Weibull Cdf is given by

$$F_S(s) = 1 - \exp\Big[-L^{\theta}\big(\frac{s}{\beta}\big)^{\alpha}\Big], \quad s > 0.$$

We also compare the fits of the proposed models with those of the power-law Weibull models.

A. Carbon Micro-Composite Tensile Strength Data

The data in this example were obtained by Bader & Priest [22] and have since then been used frequently for illustration in acceleration test literature including [1], [10], [23], [24]. The data are obtained from 1,000-filament (carbon fibers) tows impregnated with an epoxy resin, the micro-composites, which were tested in tension. The tows were tested for tensile strength at gauge lengths of 20, 50, 150 and 300 mm with 28, 30, 32 and 29 observed specimens at the respective gauge lengths. The raw data set is explicitly provided by Smith [24].

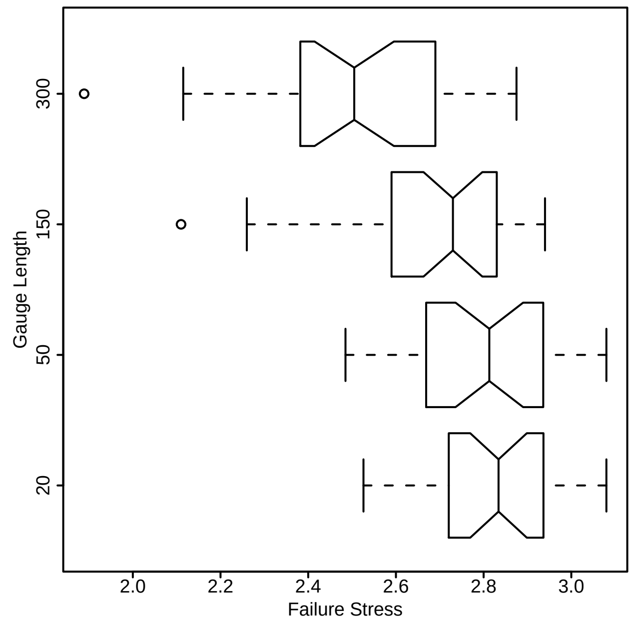


Fig. 1. Box-whisker plot.

A box-and-whisker diagram [25] of the data is presented in Fig. 1. A glance at the figure shows a tendency that breaking strengths decrease as gauge lengths increase. So the accelerated testing model seems appropriate.

We estimated the MLE of the parameters using the three models described in Section IV, and then calculated the overall MSE from the fitted models to the empirical distributions. For each model, we considered the full model with four parameters and the reduced model with three parameters (θ_3 is deleted). To help model selection, we reported the AIC. These results are summarized in Table II. The MSE of the power-law Weibull model is 8.6495×10^{-3} with the AIC being -53.1591.

One of the appropriate model selection procedures is to compare the overall MSE of the models. The four-parameter M_C has the smallest MSE. However, except the three-parameter M_A, they all give very close values. Considering the AIC criterion, the three-parameter M_B has the smallest AIC among them. Since the power-law Weibull model is based on a Weibull and the proposed models are based on an inverse Gaussian, it is not appropriate to compare the AIC of the power-law Weibull model. Note that the three-parameter M_A is the "Gauss-Gauss additive model" of Durham & Padgett [1], and Padgett [10] gave MSE $= 4.983 \times 10^{-3}$. Our resulting MSE $= 4.7993 \times 10^{-3}$ is slightly different from theirs. The difference comes from the different empirical Cdf. We used $(j - 1/2)/n_i$ for the empirical estimate of Cdf while they used $j/(n_i + 1)$. The three-parameter M_B gives a relative improvement of 70.6% in MSE compared with the power-law Weibull model and that of 47.0% compared with the Gauss-Gauss additive model. In Fig 2, we show Weibull plots of the data set. To make an obvious comparison, we superimposed fitted lines of the power-law Weibull model with the three-parameter model M_B. This also clearly shows that the model M_B fits better than the power-law Weibull model.

TABLE II

MLE, MSE AND AIC FOR MODELS UNDER CONSIDERATION

Model	$\hat{\xi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	MSE$\times 10^3$	AIC
	Four parameter model					
M_A	8.36479	23.74721	-0.04282	0.01210	2.3191	-42.5446
M_B	8.36222	23.93477	0.01020	0.00000	2.5431	-42.4955
M_C	8.30747	24.47903	0.00446	-0.00007	$\underline{2.2562}$	-41.4837
	Three parameter model					
M_A	8.29670	24.71738	0.22161		4.7993	-42.6096
M_B	8.36222	23.93477	0.01020		2.5431	$\underline{-44.4955}$
M_C	7.94546	23.45703	0.00434		2.3958	-43.0204

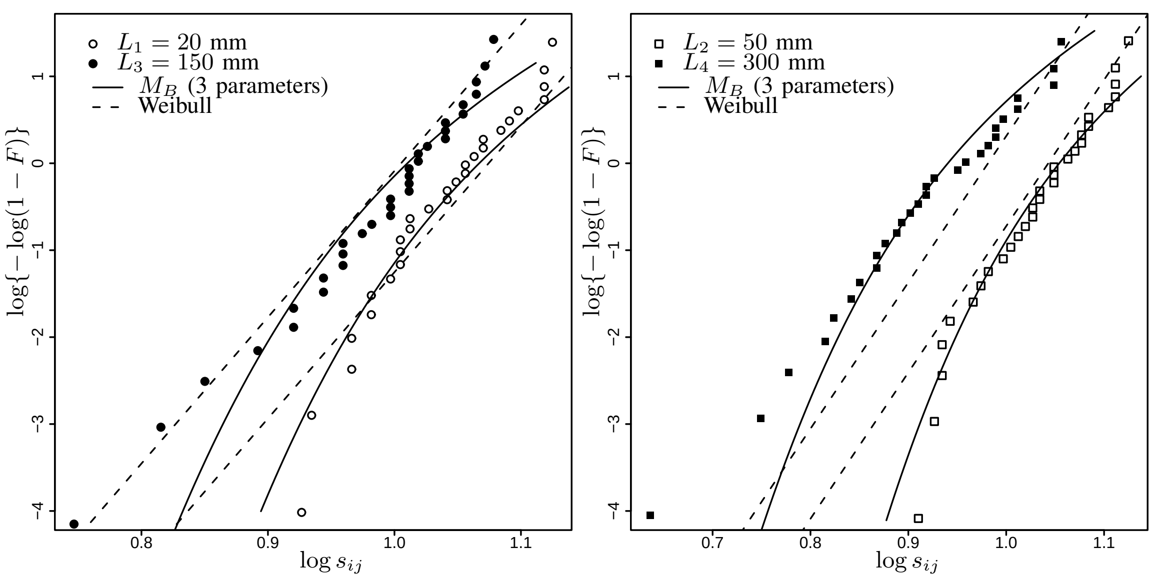


Fig. 2. Weibull probability plots (a) for the 20-mm & 150-mm tow data and (b) for the 50-mm & 300-mm tow data.

B. Carbon Fibers Embedded in Resin

The experiments in this example were conducted by B. Gul-Mohammed at the University of Surrey. Single fibers from a 1,000-fiber tow are embedded in resin. The single fibers are then mounted on special cards that expose gauge lengths of 5, 12, 30 & 75 mm with 24, 26, 25 & 26 observed specimens at the respective gauge lengths. For more details, the reader is referred to Wolstenholme [2], who also explicitly provides the raw data set.

A box-and-whisker diagram of the data shows that the accelerated testing model seems reasonable (the diagram is similar to the previous one, so not shown here).

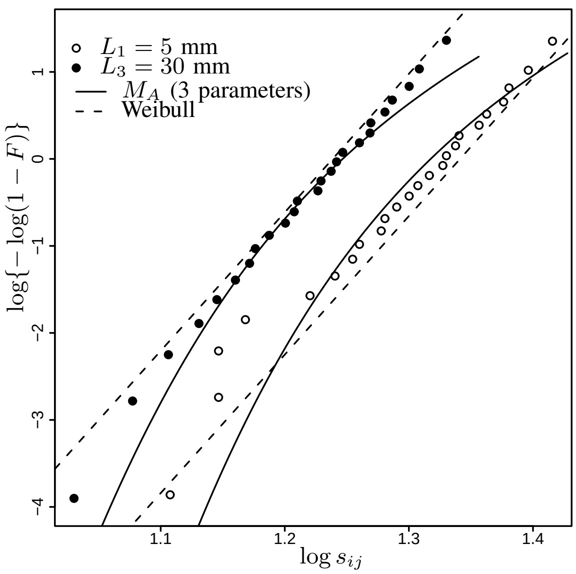
We estimated the MLE of the parameters using the three models and then calculated the overall MSE from the fitted models to the empirical distributions with the AIC reported. These results are summarized in Table III. The MSE of the power-law Weibull model is 2.2435×10^{-3} with the AIC being 12.8012.

Again, the overall MSE of the fitted models are compared. The four-parameter M_A has the smallest MSE. Considering the AIC criterion, the three-parameter M_A has the smallest value. The MSE of these two models are very close, so the reduced model is chosen. As before, it is not appropriate to compare the AIC of the power-law Weibull model with the proposed models. In Fig 3, we draw Weibull plots of the data set. We superimposed fitted lines of the power-law Weibull model with the three-parameter model M_A. This clearly shows that the model M_A fits better than the power-law Weibull model.

TABLE III

MLE, MSE and AIC for Models under Consideration

Model	$\hat{\xi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	MSE$\times 10^3$	AIC
			Four parameter model			
M_A	7.04494	27.10891	0.64997	-0.00905	<u>1.5073</u>	23.5824
M_B	6.98041	25.37618	0.04718	-0.00001	3.6896	25.3660
M_C	6.93093	33.98399	0.00432	2.18532	4.4870	26.1773
			Three parameter model			
M_A	7.04332	26.87614	0.54858		1.6231	<u>21.6287</u>
M_B	6.98041	25.37618	0.04718		3.7079	23.4253
M_C	6.82893	25.44996	0.01862		7.2150	26.9189



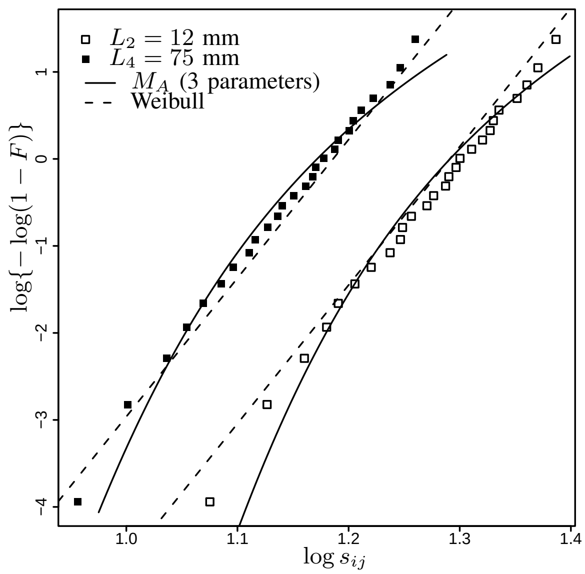


Fig. 3. Weibull probability plots (a) for the 5-mm & 30-mm tow data and (b) for the 12-mm & 75-mm tow data.

C. Polyacrylonitrile carbon fibers

Nunes, *et al.* [26] observed the tensile strength (in GPa) of individual polyacrylonitrile carbon fibers. We use some of their observations for five different gauge lengths, $L_1 = 20$mm, $L_2 = 30$mm, $L_3 = 40$mm, $L_4 = 60$mm & $L_5 = 80$mm. Their average fiber diameter was 7.2 microns measured by a laser diffraction technique. The corresponding numbers of fibers observed at the gauge lengths were $n_1 = 18$, $n_2 = 22$, $n_3 = 16$, $n_4 = 24$ & $n_5 = 14$. Each fiber was placed in an Instron universal testing machine using a load cell with tensile loading at extension rate of 0.5 mm/min. The raw data set is explicitly provided by Onar & Padgett [3].

Table IV shows the MLE of the parameters using the three models and the overall MSE from the fitted models to the empirical distributions with the AIC reported. The MSE of the power-law Weibull model is 7.2649×10^{-3} with the AIC $= 160.7814$. The MSE of the exponential-law inverse Gaussian model proposed by Onar & Padgett [3] is 8.4128×10^{-3} with the AIC $= 373.2881$. Onar & Padgett gave the MSE $= 8.2849 \times 10^{-3}$. The difference comes from the different empirical Cdf. The three-parameter model M_A (Gauss-Gauss additive model) gives the MSE $= 8.2364 \times 10^{-3}$ with the AIC $= 153.6584$. It is not appropriate to compare the AIC from different distribution families. Thus, we just compare the MSE. Among the three-parameter models, the power-law Weibull model has the smallest MSE, while among all the models considered here, the four-parameter model M_C has the smallest MSE (7.0356×10^{-3}). The four-parameter model M_C and the power-law Weibull model are fairly competitive among the models considered here.

TABLE IV

MLE, MSE AND AIC FOR MODELS UNDER CONSIDERATION

Model	$\hat{\xi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$	MSE$\times 10^3$	AIC
			Four parameter model			
M_A	2.97798	5.89321	-0.83656	0.07799	7.6106	154.7657
M_B	2.96861	8.54521	0.01603	0.00000	7.9202	155.3214
M_C	2.98283	8.16115	0.05527	-4.33587	<u>7.0356</u>	154.4688
			Three parameter model			
M_A	2.96335	9.16275	0.20710		8.2364	153.6584
M_B	2.96861	8.54521	0.01603		7.9624	153.3341
M_C	2.97489	9.44949	0.01009		7.6744	<u>153.0457</u>

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