The Hardy-Ramanujan-Rademacher Expansion of $p(n)$

Definition 0.1 *A partition of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. We denote the number of partitions of n by $p(n)$. For example, $p(4) = 5$ since the partitions of 4 are:*
4
$3+1$
$2+2$
$2+1+1$
$1+1+1+1$

The partition function has the following generating function:

$$P(q) = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

In this discussion we will focus on the exact formula for the partition function $p(n)$, given by Rademacher:

Theorem 0.2 (Rademacher)

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) k^{\frac{1}{2}} \left[\frac{d}{dx} \frac{\sinh((\pi/k)(\frac{2}{3}(x-1/24))^{\frac{1}{2}})}{(x-1/24)^{\frac{1}{2}}} \right]_{x=n}$$

where

$$A_k(n) = \sum_{h \pmod{k}, (h,k)=1} \omega_{h,k} e^{-2\pi i n h/k}$$

with $\omega_{h,k}$ a certain $24k$th root of unity defined as follows.

In order to investigate the proof of the above theorem, we need to introduce some tools.

Definition 0.3

$$F_N := \left\{ \frac{h}{k} : h, k \in \mathbb{N}, (h,k) = 1 \right\} \subset [0,1]$$

. For example,

$$F_3 = \left\{ 0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1 \right\}$$

.

Theorem 0.4 *(Farey fractions) If $\frac{h}{k}$ and $\frac{h_1}{k_1}$ are successive terms in F_N, then the rational number with the least denominator lying strictly between $\frac{h}{k}$ and $\frac{h_1}{k_1}$ is $\frac{h+h_1}{k+k_1}$ (mediant).*

Definition 0.5

$$\omega_{h,k} = \begin{cases} \left(\frac{-k}{h}\right) e^{(-\pi i(\frac{1}{4}(2-hk-h)+\frac{1}{12}(k-k^{-1})(2h-h'+h^2h'))} & \textit{if } h \textit{ odd,} \\ \left(\frac{-h}{k}\right) e^{(-\pi i(\frac{1}{4}(k-1)+\frac{1}{12}(k-k^{-1})(2h-h'+h^2h'))} & \textit{if } k \textit{ odd} \end{cases}$$

with (a/b) the Legendre symbol.

Theorem 0.6 (Knopp, 1970) *If $Rez > 0$ and h' is a solution to $hh' \equiv -1 \pmod{k}$,*

$$P(e^{2\pi i(h+iz)/k}) = \omega_{h,k} z^{\frac{1}{2}} e^{[\pi(z^{-1}-z)/12k]} P(e^{[2\pi i(h'+iz^{-1})/k]}).$$

Back to the exact formula for the partition function, let

$$P(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}.$$

Each partial product $\prod_{n=1}^{N}(1-x^n)^{-1}$ has a pole of order $\frac{N}{k}$ at $x = e^{\frac{2\pi i}{k}}$. As $z \to 0$ ($Rez > 0$), the term $e^{2\pi i(h'+iz^{-1})/k} \to 0$. By theorem 0.6,

$$P(e^{2\pi ih/k - 2\pi z/k}) \sim \omega_{h,k} z^{\frac{1}{2}} e^{\frac{\pi(z-z^{-1})}{12k}}.$$

Goal: We want to divide the circle of integration into segments depending on which "rational point" $e^{2\pi ih/k}$ we are near. We can look at the discreet set of those rational points $e^{2\pi ih/k}$ with $0 < k \leq N$, a fixed positive integer, i.e the set of proper Farey fractions of order N. The idea is we want to use the mediants as the end points for an intervals to get a natural dissection of $\mathbb{C}$. If $h_0/k_0, h/k, h_1/k_1$ are three successive terms in F_N, we write:

$$\theta'_{0,1} = \frac{1}{N+1};$$

$$\theta'_{h,k} = \frac{h}{k} - \frac{h_0+h}{k_0+h}, h > 0;$$

$$\theta''_{h,k} = \frac{h_1+h}{k_1+h} - \frac{h}{k}.$$

By the residue theorem,

$$p(n) = \frac{1}{2\pi i} \int_c \frac{P(x)}{x^{n+1}} dx.$$

Let $x = \rho e^{2\pi i\phi}$ and change the integral into polar coordinates

$$p(n) = \rho^{-n} \int_0^1 P(\rho e^{2\pi\rho})(e^{-2\pi in\phi}) d\phi$$

$$= \rho^{-n} \sum_{k=1,(h,k)=1}^{N} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} P(\rho e^{\frac{2\pi i h}{k}+2\pi i\phi}) e^{\frac{-2\pi i n h}{k}-2\pi i n\phi} d\phi.$$

Choose $\rho = e^{\frac{-2\pi}{N^2}}$.
Hence

$$p(n) = e^{\frac{2\pi n}{N^2}} \sum_{k=1,(h,k)=1}^{N} e^{\frac{-2\pi i h n}{k}} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} P(e^{\frac{2\pi i h}{k}-\frac{2\pi}{k}(\frac{k}{N^2}-ik\phi)}) e^{(-2\pi i n\phi)} d\phi.$$

Let $z = \frac{k}{N^2} - ik\phi$ and apply theorem 0.6

$$p(n) = e^{\frac{2\pi n}{N^2}} \sum_{k=1,(h,k)=1}^{N} e^{\frac{-2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\frac{\pi(z-z^{-1})}{12k}} P(e^{2\pi i(\frac{h'+iz^{-1}}{k})}) e^{-2\pi i n\phi} d\phi.$$

As $z \to 0$ with $Rez > 0$, $e^{2\pi i(h'+iz^{-1})/k} \to 0$ rapidly. Replace the integrand $P(x)$ by $(1+(P(x)-1))$ we have

$$p(n) = e^{\frac{2\pi n}{N^2}} \sum_{k=1,(h,k)=1}^{N} e^{\frac{-2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\frac{\pi(z-z^{-1})}{12k}} e^{-2\pi i n\phi} d\phi$$

$$+ e^{\frac{2\pi n}{N^2}} \sum_{k=1,(h,k)=1}^{N} e^{\frac{-2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\frac{\pi(z-z^{-1})}{12k}} P(e^{2\pi i(\frac{h'+iz^{-1}}{k})} - 1) e^{-2\pi i n\phi} d\phi.$$

Let

$$\Sigma_1 = e^{\frac{2\pi n}{N^2}} \sum_{k=1,(h,k)=1}^{N} e^{\frac{-2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\frac{\pi(z-z^{-1})}{12k}} e^{-2\pi i n\phi} d\phi$$

and

$$\Sigma_2 = e^{\frac{2\pi n}{N^2}} \sum_{k=1,(h,k)=1}^{N} e^{\frac{-2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\frac{\pi(z-z^{-1})}{12k}} P(e^{2\pi i(\frac{h'+iz^{-1}}{k})} - 1) e^{-2\pi i n\phi} d\phi.$$

We'll show the contribution of Σ_2 is negligible.
We have the bound for $z = kN^{-2} - ik\phi$:

$$|z^{\frac{1}{2}}| e^{\frac{\pi(z-z^{-1})}{12k}} P(e^{2\pi i(\frac{h'+iz^{-1}}{k})} - 1) e^{-2\pi i n\phi}| \le |z^{\frac{1}{2}}||e^{\frac{-\pi}{12N^2}} \sum_{m=1}^{\infty} p(m) e^{-2\pi Re(z^{-1})(\frac{m-1/24}{k})}.$$

We have each of $\theta'_{h,k}$ and $\theta''_{h,k}$ satisfies $\frac{1}{2kN} \le \theta_{h,k} \le \frac{1}{kN}$, $\theta'_{h,k} \le \phi \le \theta''_{h,k}$.

$$\frac{1}{k} Re(z^{-1}) = \frac{N^{-2}}{k^2(N^{-4}+\phi^2)} > \frac{N^{-2}}{k^2 N^{-4} + N^{-2}} = \frac{1}{1+k^2 N^{-2}} \ge \frac{1}{2}.$$

$$|z|^{\frac{1}{2}} = (k^2N^{-4} + k^2\phi^2)^{\frac{1}{4}} < (k^2N^{-4} + N^{-2})^{\frac{1}{4}} \le 2^{\frac{1}{4}}N^{-\frac{-1}{2}}.$$

$$|\Sigma_2| \le e^{(\frac{2\pi n}{N^2})} \sum_{k=1,(h,k)=1}^{N} 2^{\frac{1}{4}}N^{-\frac{1}{2}}e^{(-\frac{\pi}{12N^2})} \sum_{m=1}^{\infty} p(m)e^{-\pi(m-\frac{1}{24})} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} d\phi.$$

$$\le e^{(\frac{2\pi n}{N^2})}e^{(-\frac{\pi}{12N^2})}2^{\frac{1}{4}}N^{-\frac{1}{2}} \sum_{m=1}^{\infty} p(m)e^{-\pi(m-\frac{1}{24})} \sum_{k=1,(h,k)=1}^{N} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} d\phi.$$

$$\le CN^{-\frac{1}{2}}e^{\frac{2\pi n}{N^2}}.$$

This approaches to 0 as $N \to \infty$ where C is a constant.
In the integral Σ_1, we change variables to ω, where $\omega = N^{-2} - i\phi$

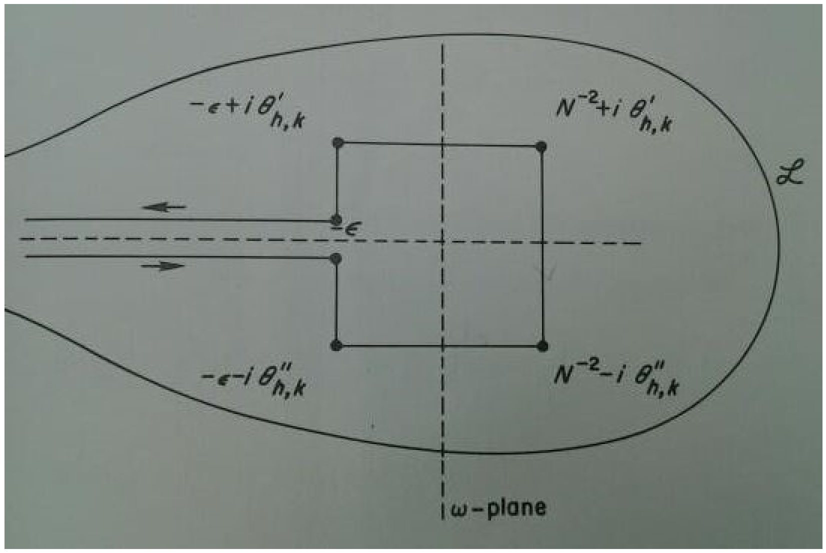
$$\Sigma_1 = e^{-\frac{2\pi n}{N^2}} \frac{k^{\frac{1}{2}}}{i} \int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} g(\omega)d\omega$$

where $g(\omega) = \omega^{\frac{1}{2}}e^{2\pi(n-\frac{1}{24})\omega+\frac{\pi}{12k^2\omega}}$.
The integrand is single valued and analytic in the complex plane, so by Cauchy's theorem we may rewrite it as

$$\Sigma_1 = e^{-\frac{2\pi n}{N^2}} \frac{k^{\frac{1}{2}}}{i}(\int_{-\infty}^{0+} - \int_{-\infty}^{-\epsilon} - \int_{-\epsilon}^{-\epsilon-i\theta''_{h,k}} - \int_{-\epsilon-i\theta''_{h,k}}^{N^{-2}-i\theta''_{h,k}} - \int_{N^{-2}+i\theta'_{h,k}}^{-\epsilon+i\theta'_{h,k}} - \int_{-\epsilon+i\theta'_{h,k}}^{-\epsilon} - \int_{-\epsilon}^{-\infty})g(\omega)d\omega.$$

$$\Sigma_1 = e^{-\frac{2\pi n}{N^2}} \frac{k^{\frac{1}{2}}}{i}(L_k - I_1 - I_2 - I_3 - I_4 - I_5 - I_6).$$

The integrals are demonstrated in the figure below, where L_k is the loop integral along the contour $\mathfrak{L}$.



I_2, I_3, I_4, I_5 can be shown to be negligible.
The integrals I_1 and I_6 are not negligible; however,

$$I_1+I_6 = \int_{-\infty}^{-\epsilon} \sqrt{|u|}e^{-\frac{\pi i}{2}} e^{\frac{\pi}{12k^2u}+2\pi(n-\frac{1}{24})u} du + \int_{-\epsilon}^{\infty} \sqrt{|u|}e^{-\frac{\pi i}{2}} e^{\frac{\pi}{12k^2u}+2\pi(n-\frac{1}{24})u} du$$

$$= -2i\int_{\epsilon}^{\infty} t^{\frac{1}{2}} e^{-2\pi(n-\frac{1}{24})t-\frac{\pi}{12k^2t}} dt$$

$$= -2iH_k.$$

Combining the results of Σ_1 and Σ_2 we get an expression for $p(n)$
Set $\psi_k(n) = \frac{k^{\frac{1}{2}}}{i}L_k + 2k^{\frac{1}{2}}H_k$, then

$$p(n) = \sum_{k=1}^{N} A_k(n)\psi_k(n) + O[N^{-\frac{1}{2}}e^{\frac{2\pi n}{N^2}}] + O(N^{-\frac{1}{2}}).$$

Note that the error terms here are all $\to 0$ as $N \to \infty$.
We state the following theorem, which completes the proof

Theorem 0.7

$$\psi_k(n) = \frac{k^{\frac{1}{2}}}{\pi\sqrt{2}} \left[\frac{d}{dx} \frac{\sinh((\pi/k)(\frac{2}{3}(x-1/24))^{\frac{1}{2}})}{(x-1/24)^{\frac{1}{2}}} \right]_{x=n}.$$