THE EQUATIONS $3X^2 - 2 = Y^2$ AND $8X^2 - 7 = Z^2$

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1. The four numbers 1, 3, 8, 120 have the property that the product of any two, increased by 1, is a perfect square. Professor J. H. van Lint, in a lecture at Oberwolfach in March 1968, discussed the problem whether there is any other positive integer that can replace 120. Since then Professor van Lint has been good enough to send us a copy of a report [1] which he gives references to the history of the problem, and also gives a proof that there is no such integer up to $10^{1700000}$.

An integer N which can replace 120 while preserving the property must have the form $N = x^2 - 1$, and the conditions are equivalent to the two equations

$$3x^2 - 2 = y^2, \quad 8x^2 - 7 = z^2 \tag{1}$$

Thus the question is whether these simultaneous equations have any solution in positive integers, other than the solutions with $x = 1$ (corresponding to $N = 0$) and $x = 11$ (corresponding to $N = 120$). The object of the present note is to prove that there is no other solution.

It is well known that two equations of the form (1) can have only finitely many solutions in integers. One way of proving this is to apply a theorem of Siegel [2] to the equation

$$(3x^2 - 2)(8x^2 - 7) = t^2.$$

Another way is to express the solutions of the separate equations in (1) by powers of quadratic irrationals, as was done by Professor van Lint, and as we shall do in §2. This leads to an inequality satisfied by a linear combination of the logarithms of three particular algebraic numbers, and to this we can apply a theorem of Gelfond [3]. Both Siegel's theorem and Gelfond's theorem depend ultimately on Thue's theorem and its refinements, and are therefore not effective. That is, they offer no possibility of determining a number X such that there is no solution with $x > X$.

(rest of article continues …)

[1] J. H. van Lint, ‘On a set of Diophantine equations’, Report 68-WSK-03 of the Technological University Eindhoven, September 1968.

[2] C. L. Siegel (under the pseudonym ‘X’), ‘The integer solutions of the equation $y^2 = ax^n + bx^{n-1} + \dots + k$,’ *J. London Math. Soc.* 1 (1926) 66–8, or *Gesammelte Abhandlungen* I, 207–8.

[3] A. O. Gelfond, *Transcendental and algebraic numbers* (Dover, New York 1960), p. 34, Theorem IV.

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