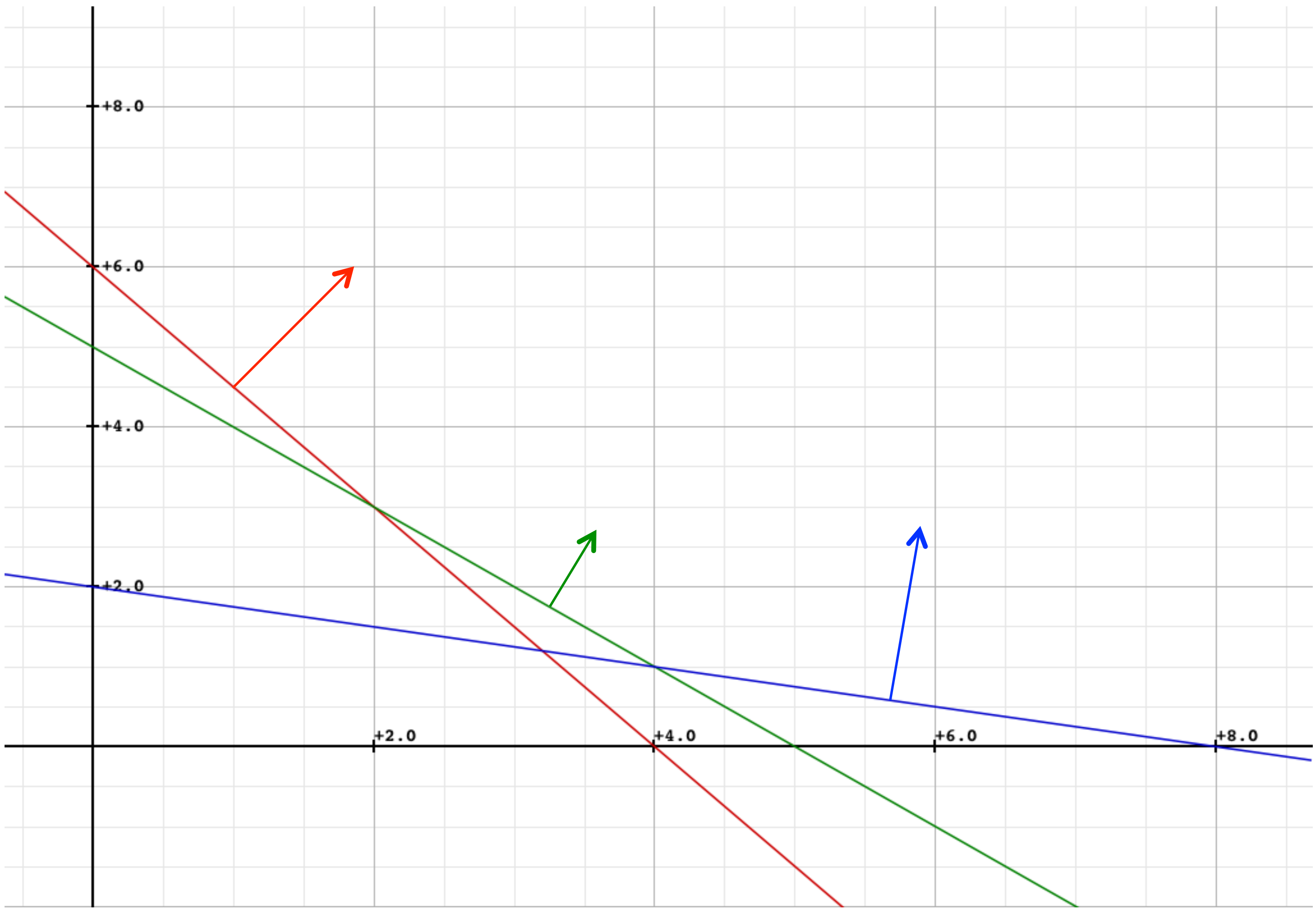
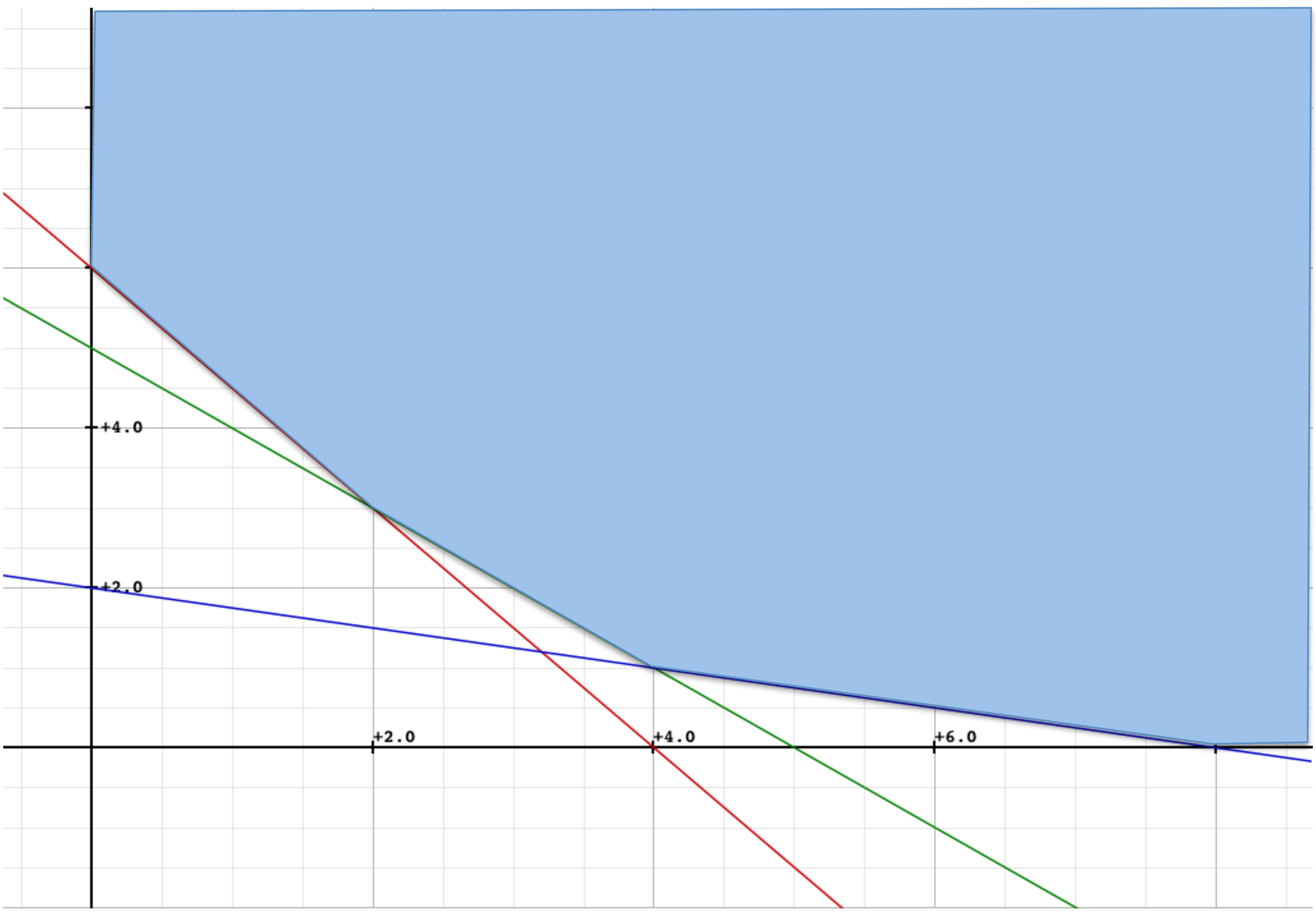
KKT Conditions

$$\begin{aligned}
\min\ & 4x_1 + 3x_2 \\
\text{s.t.}\ & 3x_1 + 2x_2 \geq 12 \\
& x_1 + x_2 \geq 5 \\
& x_1 + 4x_2 \geq 8 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}$$

$$\begin{aligned} \min\ & 4x_1 + 3x_2 \\ \text{s.t.}\ & 3x_1 + 2x_2 \geq 12 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 8 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$



$$\begin{aligned} \min \quad & 4x_1 + 3x_2 \\ \text{s.t.} \quad & 3x_1 + 2x_2 \geq 12 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 8 \\ & x_1 \geq 0, \ x_2 \geq 0 \end{aligned}$$



Is $\mathbf{x} = (4, 1)$ Optimal?

– it is *feasible*: first constraint holds with >, the other two hold with =

– $x_1 > 0, x_2 > 0, x_3 > 0$ so by CS
$u_4 = 0, u_5 = 0, u_1 = 0$

$$\begin{aligned} \min\ & 4x_1 + 3x_2 \\ \text{s.t.}\ & 3x_1 + 2x_2 \geq 12 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 8 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & 12u_1 + 5u_2 + 8u_3 \\ \text{s.t.}\ & \begin{bmatrix} 3 \\ 2 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_4 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_5 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ & u_1 \geq 0 \quad u_2 \geq 0 \quad u_3 \geq 0 \quad u_4 \geq 0 \quad u_5 \geq 0 \end{aligned}$$

Is **x** = (4, 1) Optimal?

– it is *not optimal*: the vector **c** does not lie in the cone spanned by the (inward pointing) normals of the two binding constraints.

$$
\begin{aligned}
\min\ & 4x_1 + 3x_2 \\
\text{s.t.}\ & 3x_1 + 2x_2 \geq 12 \\
& x_1 + x_2 \geq 5 \\
& x_1 + 4x_2 \geq 8 \\
& x_1 \geq 0,\ x_2 \geq 0
\end{aligned}
$$

$$
\begin{array}{lccccccccccc}
\max & 12u_1 & + & 5u_2 & + & 8u_3 & & & & & & \\
\text{s.t.} & \begin{bmatrix} 3 \\ 2 \end{bmatrix} u_1 & + & \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_2 & + & \begin{bmatrix} 1 \\ 4 \end{bmatrix} u_3 & + & \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_4 & + & \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_5 & = & \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\
& u_1 \geq 0 & & u_2 \geq 0 & & u_3 \geq 0 & & u_4 \geq 0 & & u_5 \geq 0 & &
\end{array}
$$

Is $\mathbf{x} = (2, 3)$ Optimal?

– it is *feasible*: first two constraints hold with =, the third holds with >

– $x_1 > 0, x_2 > 0, x_5 > 0$ so by CS $u_4 = 0, u_5 = 0, u_3 = 0$

$$\begin{aligned} \min\ & 4x_1 + 3x_2 \\ \text{s.t.}\ & 3x_1 + 2x_2 \geq 12 \quad \leftarrow \\ & x_1 + x_2 \geq 5 \quad \leftarrow \\ & x_1 + 4x_2 \geq 8 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$\begin{array}{llllllllllll} \max & 12u_1 & + & 5u_2 & + & 8u_3 & & & & & & \\ \text{s.t.} & \begin{bmatrix} 3 \\ 2 \end{bmatrix} u_1 & + & \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_2 & + & \begin{bmatrix} 1 \\ 4 \end{bmatrix} u_3 & + & \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_4 & + & \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_5 & = & \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ & u_1 \geq 0 & & u_2 \geq 0 & & u_3 \geq 0 & & u_4 \geq 0 & & u_5 \geq 0 & & \end{array}$$

Is **x** = (2, 3) Optimal?

– it is *optimal*: the vector **c** does lie in the cone spanned by the (inward pointing) normals of the two binding constraints.

$$\begin{aligned} \min\ & 4x_1 + 3x_2 \\ \text{s.t.}\ & 3x_1 + 2x_2 \geq 12 \\ & x_1 + x_2 \geq 5 \\ & x_1 + 4x_2 \geq 8 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \max\ & 12u_1 + 5u_2 + 8u_3 \\ \text{s.t.}\ & \begin{bmatrix} 3 \\ 2 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u_3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_4 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_5 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ & u_1 \geq 0 \quad u_2 \geq 0 \quad u_3 \geq 0 \quad u_4 \geq 0 \quad u_5 \geq 0 \end{aligned}$$