Dual Simplex Algorithm

Consider the LP:

$$\begin{array}{llllllll} \min & 2x_1 & + & 3x_2 & + & 4x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & \geq & 3 \\ & 2x_1 & - & x_2 & + & 3x_3 & \geq & 4 \\ & & & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

Place this in standard form:

$$\begin{array}{llllllllllll} \min & 2x_1 & + & 3x_2 & + & 4x_3 & + & 0x_4 & + & 0x_5 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & - & x_4 & + & 0x_5 & = & 3 \\ & 2x_1 & - & x_2 & + & 3x_3 & + & 0x_4 & - & x_5 & = & 4 \\ & & & & & x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0 \end{array}$$

Unfortunately, there is no obvious initial feasible basis B. For example choosing $B = [a_4, a_5]$ gives $x_4 = -3$, $x_5 = -4$.

However, because all c_j are nonnegative, this defines a dual feasible basis.

Namely, $B = -I$, $c_B = [0,\ 0]^T$ so that $u = [0,\ 0]^T$. This gives reduced costs $\bar{c}_j = c_j$ for all nonbasic columns a_j.

$$\bar{c}_1 = 2 \geq 0$$

$$\bar{c}_2 = 3 \geq 0$$

$$\bar{c}_3 = 4 \geq 0$$

Let a_5 leave the basis and determine which nonbasic column enters. Compute $d = [0,\ 1]$ and then calculate $d^T a_j$ for all nonbasic columns a_j.

$d^T a_1 = 2$

$d^T a_2 = -1$ $\qquad\qquad \alpha = \min\{2/2, 4/4\} = 1$, so a_1 enters.

$d^T a_3 = 3$

Now $B = [a_4, a_1]$. We can find the new x_B by solving $B x_B = b$ or simply by updating the previous $x_B = [x_4, x_5] = [-3, -4]$.

Namely find $-B^{-1}a_1 = [1 \;\; 2]$

$$x_4 = -3 + 1x_1 \geq 0$$
$$x_5 = -4 + 2x_1 \geq 0$$

Since a_5 is leaving the basis, x_5 is forced to 0, giving $x_1 = 2$, $x_4 = -1$.

Continue with $B = [a_4, a_1]$, which remains dual feasible:

$$\bar{c}_2 = 4 \geq 0$$
$$\bar{c}_3 = 1 \geq 0$$
$$\bar{c}_5 = 1 \geq 0$$

Since $x_4 = -1$, column a_4 should next leave the basis.

Compute $d = [1, -1/2]$ and then calculate $d^T a_j$ for all nonbasic columns a_j.

$d^T a_2 = 5/2$

$d^T a_3 = -1/2$ $\qquad\qquad \alpha = \min\{4/2.5, 1/.5\} = 8/5$, so a_2 enters.

$d^T a_5 = 1/2$

Now $B = [a_2, a_1]$. We can find the new x_B by updating the previous $x_B = [x_4, x_1] = [-1, 2]$. Since x_2 enters, compute $-B^{-1}a_2 = [5/2, 1/2]$:

$$
\begin{aligned}
x_4 &= -1 + (5/2)\,x_2 \geq 0 \\
x_1 &= 2 + (1/2)\,x_2 \geq 0
\end{aligned}
$$

Since a_4 is leaving the basis, x_4 is forced to 0, giving $x_2 = 2/5$ and $x_1 = 2 + (1/2)\,x_2 = 2 + (1/2)(2/5) = 11/5$.

Now $x_B = [x_2, x_1] = [2/5, 11/5] \geq 0$, so we have primal feasibility and an optimal solution. Throughout we have maintained dual feasibility and complementary slackness.