Dual Simplex Algorithm

Consider a primal LP in standard form and its dual:

$$\begin{array}{lrcl} \min & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & = & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array} \qquad \begin{array}{lrcl} \max & \mathbf{u}^T\mathbf{b} & & \\ \text{s.t.} & \mathbf{u}^T A & \leq & \mathbf{c}^T \\ & \mathbf{u} & \text{free} & \end{array} .$$

We know that a basis B of A is a primal feasible basis and $\mathbf{x}_B$ is a basic feasible primal solution if $\mathbf{x}_B = B^{-1}\mathbf{b} \geq \mathbf{0}$.

We define a vector $\mathbf{u}$ to be a **basic feasible dual solution** if $\mathbf{u}^T = \mathbf{c}_B^T B^{-1}$ and $\mathbf{u}^T A \leq \mathbf{c}^T$. In this case we say that B is a **dual feasible basis**. Of course, if B is primal feasible and dual feasible, then it is an optimal basis.

The Dual Simplex algorithm starts with a dual feasible basis B that is not primal feasible and iterates to an optimal solution. The idea is to move in the dual space from $\mathbf{u}$ along some direction $\mathbf{d}$ to the new point $\mathbf{u}' = \mathbf{u} + \alpha\mathbf{d}$, with $\alpha > 0$. This is done in such a way to preserve dual feasibility, which currently holds for all nonbasic columns $\mathbf{a}_j$: $\mathbf{u}^T\mathbf{a}_j \leq c_j$.

Since $\mathbf{u}' = \mathbf{u} + \alpha\mathbf{d}$ we require $\mathbf{u}'^T\mathbf{a}_j = \mathbf{u}^T\mathbf{a}_j + \alpha\mathbf{d}^T\mathbf{a}_j \leq c_j$. If $\mathbf{d}^T\mathbf{a}_j \leq 0$, then this imposes no restriction since $\mathbf{u}^T\mathbf{a}_j \leq c_j$. However, if $\mathbf{d}^T\mathbf{a}_j > 0$ then α must satisfy the inequality

$$\alpha \leq \frac{c_j - \mathbf{u}^T\mathbf{a}_j}{\mathbf{d}^T\mathbf{a}_j}.$$

The smallest such ratio then dictates which column $\mathbf{a}_j$ will enter the basis.

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0. Start with a dual feasible basis B. That is, $\mathbf{u}^T = \mathbf{c}_B^T B^{-1}$ and $\mathbf{u}^T A \leq \mathbf{c}^T$.

1. If $\mathbf{x}_B = B^{-1}\mathbf{b} \geq \mathbf{0}$, then B is optimal. Else, choose i^* such that $\mathbf{x}_{\beta_{i^*}} < 0$. Then column $\mathbf{a}_{\beta_{i^*}}$ *leaves* the basis.

2. Compute the direction $\mathbf{d}^T = -\boldsymbol{\varepsilon}_{i^*} B^{-1}$ which is the negative of row i^* of B^{-1}. Compute $\mathbf{d}^T\mathbf{a}_j$ for each nonbasic column $\mathbf{a}_j$. If all $\mathbf{d}^T\mathbf{a}_j \leq 0$ then the original problem is infeasible.

3. Compute the step size

$$\alpha = \min\left\{\frac{c_j - \mathbf{u}^T\mathbf{a}_j}{\mathbf{d}^T\mathbf{a}_j} : \mathbf{d}^T\mathbf{a}_j > 0\right\} = \frac{c_s - \mathbf{u}^T\mathbf{a}_s}{\mathbf{d}^T\mathbf{a}_s}$$

   Then column $\mathbf{a}_s$ *enters* the basis.

4. $\mathbf{u}' = \mathbf{u} + \alpha\mathbf{d}$, and the new basis B' is obtained by replacing column $\mathbf{a}_{\beta_{i^*}}$ with column $\mathbf{a}_s$.