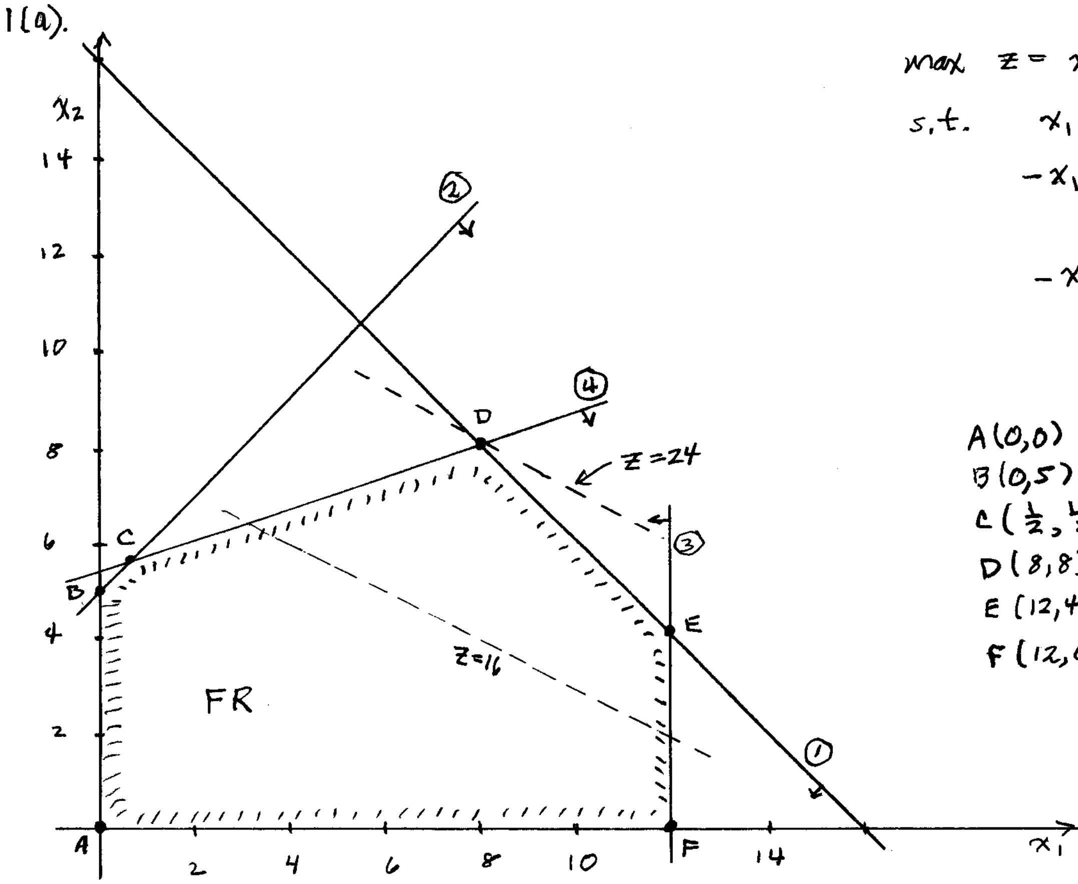
1(a).



$$\max \; z = x_1 + 2x_2$$

s.t.

$x_1 + x_2 \le 16$ ①

$-x_1 + x_2 \le 5$ ②

$x_1 \le 12$ ③

$-x_1 + 3x_2 \le 16$ ④

$x_1, x_2 \ge 0$

$A(0,0)$
$B(0,5)$
$C(\frac{1}{2}, \frac{11}{2})$
$D(8,8)$
$E(12,4)$
$F(12,0)$

} extreme points

Graphical solution: move line $x_1 + 2x_2 = c$ parallel to itself in the direction of $(1,2)$ until just about to leave the feasible region $\Rightarrow x^* = (8,8)$, $z^* = 24$.

(c) Add slacks x_3, x_4, x_5, x_6. Initial basis $\{x_3, x_4, x_5, x_6\}$

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-1	-2	0	0	0	0	0
x_3	0	1	1	1	0	0	0	16
x_4	0	-1	1	0	1	0	0	5
x_5	0	1	0	0	0	1	0	12
x_6	0	-1	3	0	0	0	1	16

$x = (0,0,16,5,12,16)$

$z = 0$ Ⓐ

x_2 enters

x_4 leaves

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-3	0	0	2	0	0	10
x_3	0	2	0	1	-1	0	0	11
x_2	0	-1	1	0	1	0	0	5
x_5	0	1	0	0	0	1	0	12
x_6	0	2	0	0	-3	0	1	1

$x=(0,5,11,0,12,1)$
$z=10$ (B)
x_1 enters
x_6 leaves

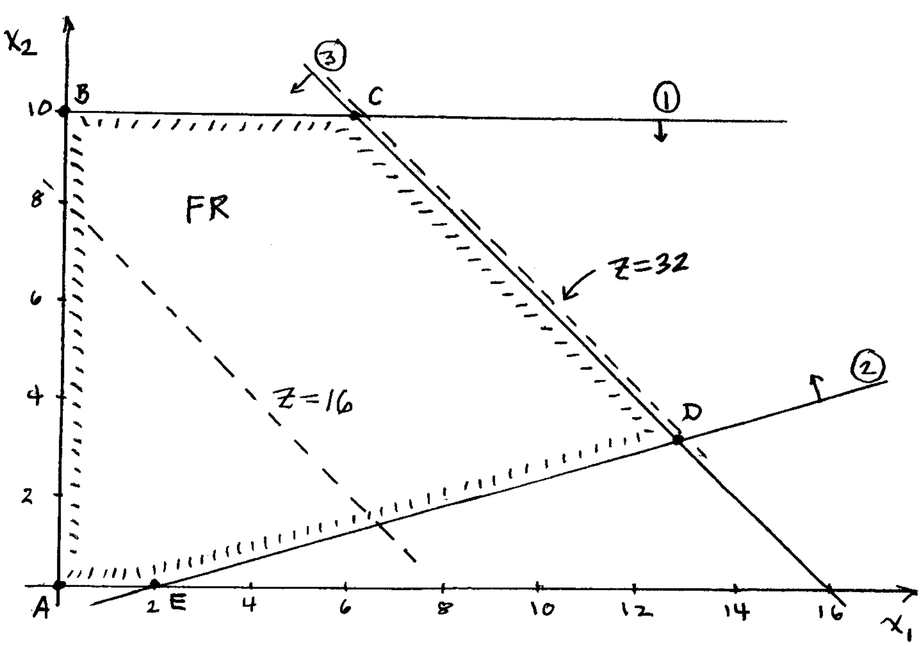
	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	$-\frac{5}{2}$	0	$\frac{3}{2}$	$\frac{23}{2}$
x_3	0	0	0	1	2	0	-1	10
x_2	0	0	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{11}{2}$
x_5	0	0	0	0	$\frac{3}{2}$	1	$-\frac{1}{2}$	$\frac{23}{2}$
x_1	0	1	0	0	$-\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$

$x=(\frac{1}{2},\frac{11}{2},10,0,\frac{23}{2},0)$
$z=\frac{23}{2}$ (C)
x_4 enters
x_3 leaves

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	$\frac{5}{4}$	0	0	$\frac{1}{4}$	24
x_4	0	0	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	5
x_2	0	0	1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	8
x_5	0	0	0	$-\frac{3}{4}$	0	1	$\frac{1}{4}$	4
x_1	0	1	0	$\frac{3}{4}$	0	0	$-\frac{1}{4}$	8

$x^*=(8,8,0,5,4,0)$
$z^*=24$ (D)
optimal since all reduced costs ≥ 0

2(a).



$$\max\ z = 2x_1 + 2x_2$$

s.t.
$x_2 \leq 10$ ①
$x_1 - 3x_2 \leq 2$ ②
$x_1 + x_2 \leq 16$ ③
$x_1, x_2 \geq 0$

A(0,0)
B(0,10)
C(6,10)
D($\frac{25}{2}$, $\frac{7}{2}$)
E(2,0)
} extreme points

Graphically, all points along the line segment joining Ⓒ & Ⓓ are optimal: alternative optimal solutions, with $z^*=32$

2(b) Add slacks x_3, x_4, x_5. Initial basis $\{x_3, x_4, x_5\}$

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	-2	0	0	0	0
x_3	0	0	1	1	0	0	10
x_4	0	1	-3	0	1	0	2
x_5	0	1	1	0	0	1	16
z	1	0	-8	0	2	0	4
x_3	0	0	1	1	0	0	10
x_1	0	1	-3	0	1	0	2
x_5	0	0	4	0	-1	1	14
z	1	0	0	0	0	2	32
x_3	0	0	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{13}{2}$
x_1	0	1	0	0	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{25}{2}$
x_2	0	0	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{7}{2}$
z	1	0	0	0	0	2	32
x_4	0	0	0	4	1	-1	26
x_1	0	1	0	-1	0	1	6
x_2	0	0	1	1	0	0	10

$x = (0, 0, 10, 2, 16)$
$z = 0$
x_1 enters
x_4 leaves

$x = (2, 0, 10, 0, 14)$
$z = 4$
x_2 enters
x_5 leaves

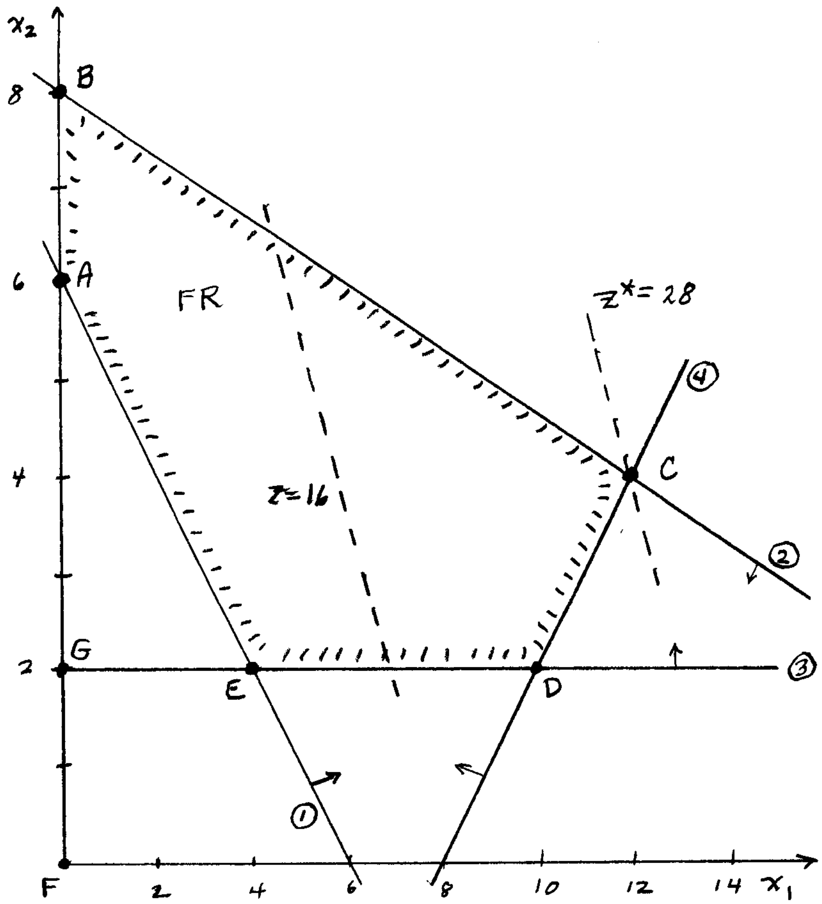
$x^* = (\frac{25}{2}, \frac{7}{2}, \frac{13}{2}, 0, 0)$
$z^* = 32$
optimal since all red. costs ≥ 0.
NOTE: $z_4 - c_4 = 0$, so may have alternative opt. soln. Let x_4 enter; x_3 leaves

$x^* = (6, 10, 0, 26, 0)$
$z^* = 32$

All opt. solns: $\lambda(\frac{25}{2}, \frac{7}{2}, \frac{13}{2}, 0, 0) + (1-\lambda)(6, 10, 0, 26, 0)$, $0 \leq \lambda \leq 1$.

e.g. $\lambda = \frac{1}{2}$ $\Rightarrow$ $x^* = (\frac{37}{4}, \frac{27}{4}, \frac{13}{4}, 13, 0)$, $z^* = 32$ is optimal but is not a BFS.

3(a).



max $z = 2x_1 + x_2$

s.t. $x_1 + x_2 \geq 6$ ①

$x_1 + 3x_2 \leq 24$ ②

$x_2 \geq 2$ ③

$x_1 - x_2 \leq 8$ ④

$x_1, x_2 \geq 0$

A(0,6)
B(0,8)
C(12,4)
D(10,2)
E(4,2)

} extreme points

Move the obj. function line $2x_1 + x_2 = c$ parallel to itself in the direction of (2,1) until it is about to leave the FR $\Rightarrow$ $x^* = (12,4)$, $z^* = 28$.

(b) Phase I: max $-x_7 - x_8$ ← artificial variables

s.t.

$$\begin{aligned} x_1 + x_2 - x_3 + x_7 &= 6 \\ x_1 + 3x_2 + x_4 &= 24 \\ x_2 - x_5 + x_8 &= 2 \\ x_1 - x_2 + x_6 &= 8 \\ x_1, x_2, \dots, x_8 &\geq 0 \end{aligned}$$

Initial basis $\{x_7, x_4, x_8, x_6\}$

$$\begin{aligned} z &= -x_7 - x_8 \\ &= -[6 - x_1 - x_2 + x_3] - [2 - x_2 + x_5] \\ &= -8 + x_1 + 2x_2 - x_3 - x_5 \end{aligned}$$

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	-1	-2	1	0	1	0	0	0	-8
x_7	0	1	1	-1	0	0	0	1	0	6
x_4	0	1	3	0	1	0	0	0	0	24
x_8	0	0	1	0	0	-1	0	0	1	2
x_6	0	1	-1	0	0	0	1	0	0	8

$x = (0,0,0,24,0,8,6,2)$

$z = -8$ F

x_2 enters

x_8 leaves

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	-1	0	1	0	-1	0	0	2	-4
x_7	0	1	0	-1	0	1	0	1	-1	4
x_4	0	1	0	0	1	3	0	0	-3	18
x_2	0	0	1	0	0	-1	0	0	1	2
x_6	0	1	0	0	0	-1	1	0	1	10

$x = (0,2,0,18,0,10,4,0)$

$z = -4$ G

x_1 enters

x_7 leaves

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	0	0	0	0	0	0	1	1	0
x_1	0	1	0	-1	0	1	0	1	-1	4
x_4	0	0	0	1	1	2	0	-1	-2	14
x_2	0	0	1	0	0	-1	0	0	1	2
x_6	0	0	0	1	0	-2	1	-1	2	6

$x = (4,2,0,14,0,6,0,0)$

$z = 0$ E

optimal for Phase I (x_7, x_8 forced out of basis)

Phase II:

Drop x_7, x_8; nonbasic vars $\{x_3, x_5\}$. Original objective function added

$$z = 2x_1 + x_2 = 2[4 + x_3 - x_5] + [2 + x_5] = 10 + 2x_3 - x_5$$

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	-2	0	1	0	10
x_1	0	1	0	-1	0	1	0	4
x_4	0	0	0	1	1	2	0	14
x_2	0	0	1	0	0	-1	0	2
x_6	0	0	0	1	0	-2	1	6

$x = (4,2,0,14,0,6)$ E

$z = 10$

x_3 enters

x_6 leaves

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	0	-3	2	22
x_1	0	1	0	0	0	-1	1	10
x_4	0	0	0	0	1	4	-1	8
x_2	0	0	1	0	0	-1	0	2
x_3	0	0	0	1	0	-2	1	6

$x = (10,2,6,8,0,0)$

$z = 22$ D

x_5 enters

x_4 leaves

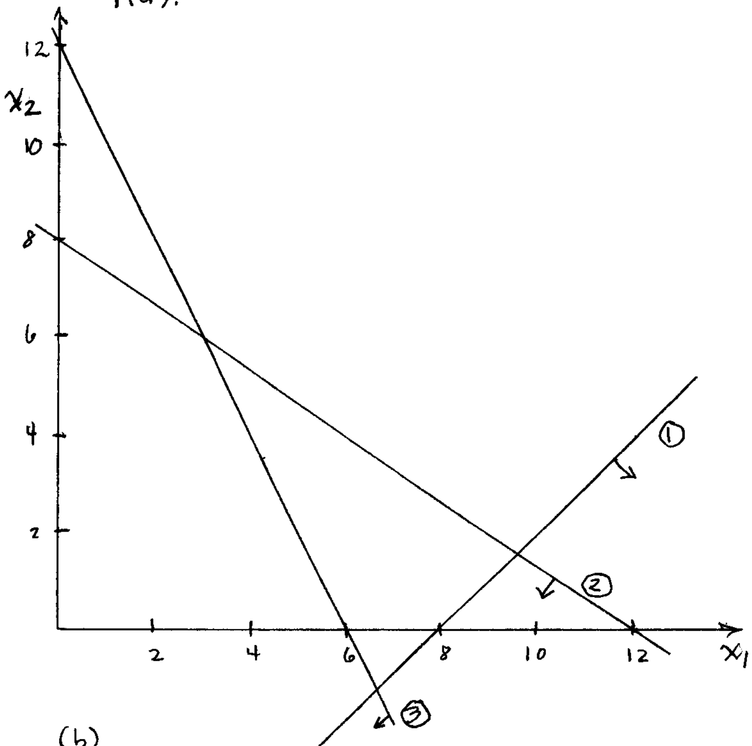
	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	$\frac{3}{4}$	0	$\frac{5}{4}$	28
x_1	0	1	0	0	$\frac{1}{4}$	0	$\frac{3}{4}$	12
x_5	0	0	0	0	$\frac{1}{4}$	1	$-\frac{1}{4}$	2
x_2	0	0	1	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	4
x_3	0	0	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	10

$x^* = (12,4,10,0,2,0)$ C

$z^* = 28$

optimal

4(a).



max $z = 2x_1 + x_2$

s.t. $x_1 - x_2 \geq 8$ (1)

$2x_1 + 3x_2 \leq 24$ (2)

$2x_1 + x_2 \leq 12$ (3)

$x_1, x_2 \geq 0$

The problem is infeasible, since no points satisfy all constraints (including nonnegativity).

(b)

Phase I: max $-x_6$

s.t. $x_1 - x_2 - x_3 + x_6 = 8$ (artificial: x_6)

$2x_1 + 3x_2 + x_4 = 24$

$2x_1 + x_2 + x_5 = 12$

$x_1, x_2, \ldots, x_5 \geq 0$

Initial basis $\{x_6, x_4, x_5\}$

$z = -x_6 = -[8 - x_1 + x_2 + x_3]$

$= -8 + x_1 - x_2 - x_3$

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-1	1	1	0	0	0	-8
x_6	0	1	-1	-1	0	0	1	8
x_4	0	2	3	0	1	0	0	24
x_5	0	2	1	0	0	1	0	12
z	1	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	-2
x_6	0	0	$-\frac{3}{2}$	-1	0	$-\frac{1}{2}$	1	2
x_4	0	0	2	0	1	-1	0	12
x_1	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	6

all red. costs ≥ 0 so optimal for Phase I. However $z^* = -2 \neq 0$ ($x_6^* = 2 > 0$), so the original problem is INFEASIBLE.