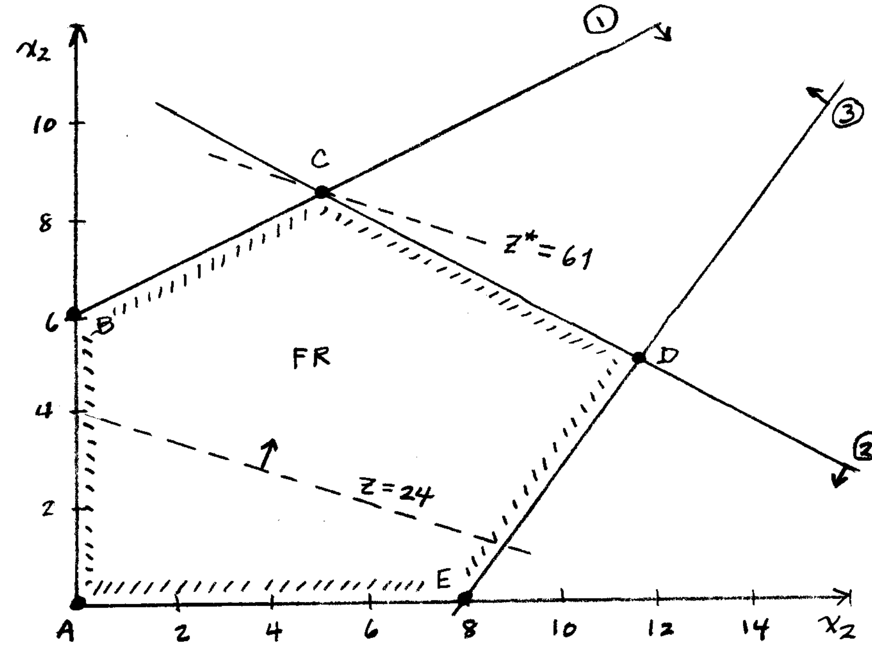
1(a).



$$\max \ z = 2x_1 + 6x_2$$

s.t.

$$-x_1 + 2x_2 \le 12 \quad ①$$
$$x_1 + 2x_2 \le 22 \quad ②$$
$$x_1 - x_2 \le 8 \quad ③$$
$$x_1, x_2 \ge 0$$

$\left.\begin{array}{l} A(0,0) \\ B(0,6) \\ C(5, \frac{17}{2}) \\ D(\frac{38}{3}, \frac{14}{3}) \\ E(8,0) \end{array}\right\}$ extreme points

Move objective function parallel to itself in the direction $(2,6) \Rightarrow$ point C.
Optimal solution is $x^* = (5, \frac{17}{2})$ with $z^* = 61$.

(c) Add slacks x_3, x_4, x_5; these are the initial basic variables with $B = I$.

$c_B = 0$; $\pi = c_B B^{-1} = 0$

$z_1 - c_1 = \pi a_1 - c_1 = -2$

$z_2 - c_2 = \pi a_2 - c_2 = -6 \leftarrow$

so x_2 enters; then

$\alpha_2 = B^{-1} a_2 = a_2$. x_3 leaves

$\pi = [3\ 0\ 0]$

$z_1 - c_1 = \pi a_1 - c_1 = -5 \leftarrow$

$z_3 - c_3 = \pi a_3 - c_3 = 3$

so x_1 enters; then $\alpha_1 = B^{-1} a_1 = [-\frac{1}{2}, 2, \frac{1}{2}]^T$. x_4 leaves

---

$\pi = [\frac{1}{2}\ \frac{5}{2}\ 0]$

$z_3 - c_3 = \pi a_3 - c_3 = \frac{1}{2} \ge 0$

$z_4 - c_4 = \pi a_4 - c_4 = \frac{5}{2} \ge 0$

z	0	0	0	0	-6	$x = (0,0,12,22,8)$, $z = 0$
x_3	1	0	0	12	2	$\leftarrow \alpha_2$
x_4	0	1	0	22	2	A
x_5	0	0	1	8	-1	
z	3	0	0	36	-5	$x = (0,6,0,10,14)$, $z = 36$
x_2	$\frac{1}{2}$	0	0	6	$-\frac{1}{2}$	
x_4	-1	1	0	10	2	α_1 B
x_5	$\frac{1}{2}$	0	1	14	$\frac{1}{2}$	
z	$\frac{1}{2}$	$\frac{5}{2}$	0	61		$x^* = (5, \frac{17}{2}, 0, 0, \frac{23}{2})$, $z^* = 61$
x_2	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{17}{2}$		optimal
x_1	$-\frac{1}{2}$	$\frac{1}{2}$	0	5		(all reduced costs ≥ 0)
x_5	$\frac{3}{4}$	$-\frac{1}{4}$	1	$\frac{23}{2}$		C

2(a)

$$A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{bmatrix} 20 \\ 12 \\ 30 \end{bmatrix} = b$$

$$c = \begin{bmatrix} 1 & 2 & 5 & 1 & -1 & 1 & -1 \end{bmatrix}$$

this is feasible: $Ax = b, x \geq 0$

$x = (10, 0, 16, 6, 0, 0, 0)$: basic vars $\{x_1, x_3, x_4\}$, nonbasic vars $\{x_2, x_5, x_6, x_7\}$

$$B = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \qquad \pi = c_B B^{-1}; \quad \pi B = c_B; \qquad \begin{aligned} \pi_1 - \pi_2 + 2\pi_3 &= 1 \\ \pi_1 + \pi_2 + \pi_3 &= 5 \\ -\pi_1 + \pi_2 - \pi_3 &= 1 \end{aligned}$$

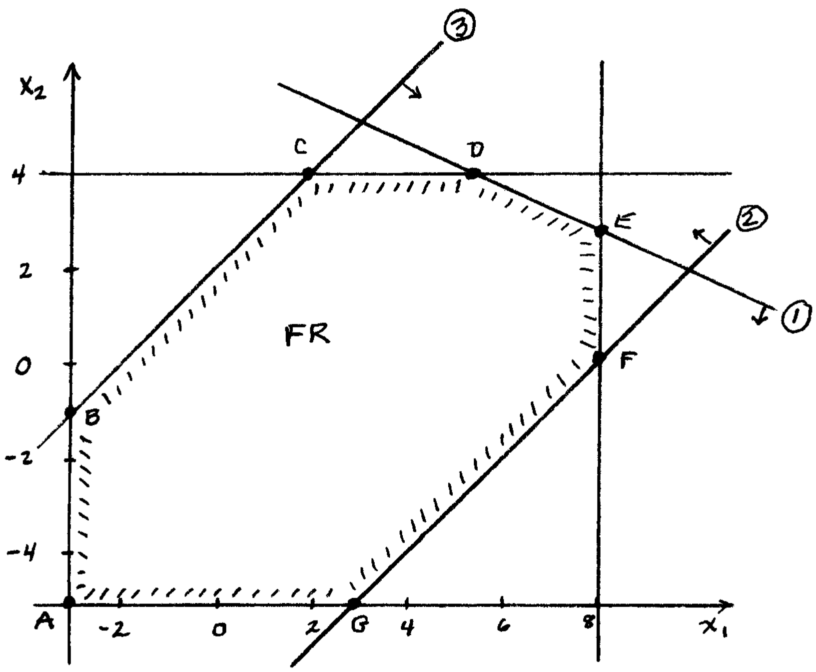
$$\Rightarrow \pi = (0\ 3\ 2)$$

$$c_B = \begin{bmatrix} 1 & 5 & 1 \end{bmatrix}$$

$$\left. \begin{aligned} z_2 - c_2 &= \pi a_2 - c_2 = 5 - 2 = 3 \\ z_5 - c_5 &= \pi a_5 - c_5 = 0 - (-1) = 1 \\ z_6 - c_6 &= \pi a_6 - c_6 = 3 - 1 = 2 \\ z_7 - c_7 &= \pi a_7 - c_7 = 2 - (-1) = 3 \end{aligned} \right\} \text{ since all reduced costs } \geq 0 \Rightarrow \underline{\text{optimal}}$$

(b) Unique, since all reduced costs of nonbasic variables are > 0.

3(a)



$$\max \ z = x_1 + 4x_2$$

s.t.

$$\begin{aligned} x_1 + 2x_2 &\leq 13 \quad ① \\ x_1 - x_2 &\leq 8 \quad ② \\ -x_1 + x_2 &\leq 2 \quad ③ \\ -3 \leq x_1 &\leq 8 \\ -5 \leq x_2 &\leq 4 \end{aligned}$$

$$\left. \begin{aligned} &A(-3,-5) \\ &B(-3,-1) \\ &C(2,4) \\ &D(5,4) \\ &E(8, \tfrac{5}{2}) \\ &F(8,0) \\ &G(3,-5) \end{aligned} \right\} \text{extreme points}$$

(b) Add slacks: max $z = x_1 + 4x_2$

s.t. $x_1 + 2x_2 + x_3 = 13$

$x_1 - x_2 + x_4 = 8$

$-x_1 + x_2 + x_5 = 2$

$-3 \le x_1 \le 8, \; -5 \le x_2 \le 4, \quad 0 \le x_3, x_4, x_5$

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$C = [1 \;\; 4 \;\; 0 \;\; 0 \;\; 0]$

Extreme Pt	NB at lower bd	NB at upper bd	Basic
A	x_1, x_2		x_3, x_4, x_5
B	x_1, x_5		x_2, x_3, x_4
C	x_5	x_2	x_1, x_3, x_4
D	x_3	x_2	x_1, x_4, x_5
E	x_3	x_1	x_2, x_4, x_5
F	x_4	x_1	x_2, x_3, x_5
G	x_2, x_4		x_1, x_3, x_5

(c) x_1, x_2 at LB: $x = (\overset{LB}{-3}, \overset{LB}{-5}, 26, 6, 4)$; $B = I$, $c_B = [0\;0\;0]$

$\pi B = c_B$, $\pi = [0\;0\;0]$; $z_1 - c_1 = \pi a_1 - c_1 = 0 - 1 = -1 < 0$ (LB)

$z = -23$ $\qquad z_2 - c_2 = \pi a_2 - c_2 = 0 - 4 = -4 < 0$ (LB)

so x_2 enters, increases from LB; x_1 stays at -3

$x_B = B^{-1}b - B^{-1}N x_N \Rightarrow$

$$\left.\begin{aligned} x_3 &= 13 - x_1 - 2x_2 = 16 - 2x_2 \ge 0 \Rightarrow x_2 \le 8 \\ x_4 &= 8 - x_1 + x_2 = 11 + x_2 \\ x_5 &= 2 + x_1 - x_2 = -1 - x_2 \ge 0 \Rightarrow x_2 \le -1 \end{aligned}\right\} x_2 = -1$$

also $x_2 \le 4$

$\therefore$ x_5 leaves (forced to 0, its LB) $\qquad x = (\overset{LB}{-3}, -1, 18, 10, \overset{LB}{0})$, $z = -7$

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad c_B = [4\;0\;0]; \quad \pi B = c_B: \begin{aligned} 2\pi_1 - \pi_2 + \pi_3 &= 4 \\ \pi_1 &= 0 \\ \pi_2 &= 0 \end{aligned} \Rightarrow \pi = (0\;0\;4)$$

LB: $z_1 - c_1 = \pi a_1 - c_1 = -4 - 1 = -5 < 0$ $\Rightarrow$ so x_1 enters, increases from LB

{ x_5 stays at 0 }

LB: $z_5 - c_5 = \pi a_5 - c_5 = 4 - 0 = 4 \ge 0$ ✓

$\alpha_1 = B^{-1}a_1$; $\quad B\alpha_1 = a_1$; $\quad \begin{aligned} 2y_1 + y_2 &= 1 \\ -y_1 + y_3 &= 1 \\ y_1 &= -1 \end{aligned} \Rightarrow \alpha_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

$\alpha_1 = (y_1\, y_2\, y_3)^T$

$$\alpha_5 = B^{-1}a_5; \quad B\alpha_5 = a_5: \quad \begin{aligned} 2y_1 + y_2 &= 0 \\ -y_1 + y_3 &= 0 \\ y_1 &= 1 \end{aligned} \Rightarrow \alpha_5 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\beta = B^{-1}b; \quad B\beta = b: \quad \begin{aligned} 2\beta_1 + \beta_2 &= 13 \\ -\beta_1 + \beta_3 &= 8 \\ \beta_1 &= 2 \end{aligned} \Rightarrow \beta = \begin{bmatrix} 2 \\ 9 \\ 10 \end{bmatrix}$$

$x_B = B^{-1}b - B^{-1}Nx_N \Rightarrow$

$\{x_1$ increases; $x_5 = 0\}$

$$\left.\begin{aligned} x_2 &= 2 + x_1 - x_5 = 2 + x_1 \le 4 \Rightarrow x_1 \le 2 \\ x_3 &= 9 - 3x_1 + 2x_5 = 9 - 3x_1 \ge 0 \Rightarrow x_1 \le 3 \\ x_4 &= 10 \qquad - x_5 = 10 \ge 0 \\ &\text{also } x_1 \le 8 \end{aligned}\right\} \quad x_1 = 2$$

$\therefore$ x_2 leaves (forced to UB) $\quad x = (2, \overset{UB}{4}, 3, 10, \overset{LB}{0})$, $z = 18$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \quad c_B = [1 \ 0 \ 0]; \quad \pi B = c_B: \quad \left.\begin{aligned} \pi_1 + \pi_2 - \pi_3 &= 1 \\ \pi_1 &= 0 \\ \pi_2 &= 0 \end{aligned}\right\} \Rightarrow \pi = [0 \ 0 \ -1]$$

(UB) $z_2 - c_2 = \pi a_2 - c_2 = -5 \le 0$ ✓

(LB) $z_5 - c_5 = \pi a_5 - c_5 = -1 \Rightarrow$ so x_5 enters, increases from LB $\{x_2$ stays at $4\}$

Solving for $\alpha_2, \alpha_5, \beta$ we get $x_B = B^{-1}b - B^{-1}Nx_N$

$$\left.\begin{aligned} x_1 &= -2 + x_2 + x_5 = 2 + x_5 \le 8 \Rightarrow x_5 \le 6 \\ x_3 &= 15 - 3x_2 - x_5 = 3 - x_5 \ge 0 \Rightarrow x_5 \le 3 \\ x_4 &= 10 \qquad - x_5 = 10 - x_5 \ge 0 \Rightarrow x_5 \le 10 \\ &\text{also } x_5 \ge 0 \end{aligned}\right\}$$

$x_5 = 3$, and x_3 leaves (forced to $\overset{}{\underset{LB}{0}}$)

$x = (5, \overset{UB}{4}, \overset{LB}{0}, 7, 3)$, $z = 21$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad c_B = [1 \ 0 \ 0]; \quad \pi B = c_B: \quad \left.\begin{aligned} \pi_1 + \pi_2 - \pi_3 &= 1 \\ \pi_2 &= 0 \\ \pi_3 &= 0 \end{aligned}\right\} \Rightarrow \pi = [1 \ 0 \ 0]$$

$$\left.\begin{aligned} &\text{(UB)} \quad z_2 - c_2 = \pi a_2 - c_2 = -2 \le 0 \\ &\text{(LB)} \quad z_3 - c_3 = \pi a_3 - c_3 = 1 \ge 0 \end{aligned}\right\}$$ reduced costs of correct sign, so optimal

$x^* = (5, 4, 0, 7, 3)$, $z^* = 21$

(d) Here we followed the path of extreme points: A → B → C → D