Feasibility and Optimality

Feasibility:

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_N.$$

Selecting $\mathbf{x}_N = \mathbf{0}$ gives the **basic solution**

$$\begin{bmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{bmatrix} = \begin{bmatrix} B^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}.$$

If $B^{-1}\mathbf{b} \geq \mathbf{0}$, this defines a **basic feasible solution**.

Optimality:

$$z = \mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}N)\mathbf{x}_N.$$

If $(\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}N) \geq \mathbf{0}$, we have an **optimal solution**.

For the casting example, $\mathbf{c}^T = [28, 30, 20, 25, 0, 0]$ with the constraint matrix A and the right hand side vector $\mathbf{b}$ given by

$$A = \begin{bmatrix} 3 & 2 & 1 & 4 & 1 & 0 \\ 4 & 3 & 3 & 3 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix}.$$

Consider the basis $B = [\mathbf{a}_3, \mathbf{a}_5, \mathbf{a}_6]$. Thus

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \; B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \mathbf{x}_B = B^{-1}\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix},$$

so this defines a feasible basis.

Express the basic variables in terms of the nonbasic variables:

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_N$$

or,

$$\begin{array}{rcrrrrl} x_3 & = & 3 & -x_1 & -x_2 & -x_4 & \geq 0 \\ x_5 & = & 4 & -2x_1 & -x_2 & -3x_4 & \geq 0 \\ x_6 & = & 1 & +x_1 & & & \geq 0 \end{array}$$

These inequalities can be rewritten as

$$\begin{array}{rcrcrl} x_1 & + & x_2 & + & x_4 & \leq 3 \\ 2x_1 & + & x_2 & + & 3x_4 & \leq 4 \\ -x_1 & & & & & \leq 1 \end{array}$$

with x_3, x_5, x_6 acting as slack variables for these inequalities. Notice that the origin $x_1 = 0, x_2 = 0, x_4 = 0$ is feasible to this inequality system. Moreover we can express the objective function in terms of the nonbasic variables using

$$z = \mathbf{c}^T\mathbf{x} = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} N)\mathbf{x}_N.$$

Here $\mathbf{c}_B^T = [c_3, c_5, c_6] = [20, 0, 0]$, $\mathbf{c}_N^T = [c_1, c_2, c_4] = [28, 30, 25]$ so that $\mathbf{c}_B^T B^{-1} = [0, 0, 20]$. Also, $N = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4]$. Substituting into the above expression gives

$$z = \mathbf{c}_B^T B^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1} N)\mathbf{x}_N = 60 + 8x_1 + 10x_2 + 5x_4$$

In other words, we have transformed the original system into the equivalent inequality system

$$\begin{array}{lrcrcrl} \min & 8x_1 & + & 10x_2 & + & 5x_4 & \\ \text{s.t.} & x_1 & + & x_2 & + & x_4 & \leq 3 \\ & 2x_1 & + & x_2 & + & 3x_4 & \leq 4 \\ & -x_1 & & & & & \leq 1 \\ & x_1 \geq 0 & , & x_2 \geq 0 & , & x_4 \geq 0 & \end{array}$$

where the new origin $x_1 = 0, x_2 = 0, x_4 = 0$ is not only feasible but clearly optimal!