Test #2 – MthSc 440

① original problem

$$A = \begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad b = \begin{bmatrix} 19 \\ 11 \\ 29 \end{bmatrix}$$

$$c = [\,1 \quad 5 \quad 11 \quad 1 \quad 0 \quad 0 \quad 0\,]$$

(a) current Basis $\{x_2, x_1, x_7\}$

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & \frac{1}{2} & 1 \end{bmatrix}$$ ← from columns of $\{x_5, x_6, x_7\}$ in current tableau

Then $B^{-1}b = \begin{bmatrix} 15 \\ 4 \\ 6 \end{bmatrix}$ f; $\quad \pi = c_B B^{-1} = [\,5 \; 1 \; 0\,]\, B^{-1} = \begin{array}{c} \begin{array}{ccc} a & b & \end{array} \\ [\,3 \quad 2 \quad 0\,] \end{array}$

Transformed a_4: $\quad \alpha_4 = B^{-1}a_4 = B^{-1}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{array}{c} \\ d \\ e \end{array}$

$z = x_1 + 5x_2 + 11x_3 + x_4 = 4 + 5(15) = 79$ c

(b) $x = (4, 15, 0, 0, 0, 0, 6)$

(c) x_3 enters (red cost $= -3$); $\quad \min\left\{ \overset{x_2}{\frac{15}{3/2}}, \overset{x_1}{\frac{4}{1/2}} \right\} = \min\{10, 8\}$, so x_1 leaves.

$x_3 \uparrow 8$

From current tableau

$$\left.\begin{aligned} x_2 + \tfrac{3}{2}x_3 + \tfrac{1}{2}x_5 + \tfrac{1}{2}x_6 &= 15 \\ x_1 + \tfrac{1}{2}x_3 - x_4 + \tfrac{1}{2}x_5 - \tfrac{1}{2}x_6 &= 4 \\ x_7 - \tfrac{3}{2}x_3 + x_4 - \tfrac{3}{2}x_5 + \tfrac{1}{2}x_6 &= 6 \end{aligned}\right\} \qquad \begin{aligned} x_2 &= 15 - \tfrac{3}{2}x_3 = 15 - \tfrac{3}{2}(8) = 3 \\ x_1 &= 4 - \tfrac{1}{2}x_3 = 4 - \tfrac{1}{2}(8) = 0 \\ x_7 &= 6 + \tfrac{3}{2}x_3 = 6 + \tfrac{3}{2}(8) = 18 \end{aligned}$$

$\Rightarrow x = (0, 3, 8, 0, 0, 0, 18), \quad z = 103$

② problem
$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 2 & 3 & -1 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 19 \\ 27 \end{bmatrix}$$

$$c = [\,5 \quad 9 \quad 1 \quad 0 \quad 0\,]$$

$0 \le x_1 \le 15$
$1 \le x_2 \le 6$ $\qquad 0 \le x_4$
$1 \le x_3 \le 3$ $\qquad 0 \le x_5$

(a) x_1, x_2 basic $\Rightarrow x_3, x_4, x_5$ nonbasic [set to UB/LB]

- $x_3 = 1, x_4 = 0, x_5 = 0$: $\quad x_1 + 2x_2 + x_3 + x_4 = 19 \Rightarrow x_1 + 2x_2 = 18$
  $2x_1 + 3x_2 - x_3 + x_5 = 27 \qquad 2x_1 + 3x_2 = 28$
  $\Rightarrow x_1 = 2,\ x_2 = 8 \Rightarrow x = (2, 8, 1, 0, 0)$ unfeasible since $x_2 > 6$

- $x_3 = 3, x_4 = 0, x_5 = 0$: $\quad x_1 + 2x_2 = 16 \Rightarrow x_1 = 12,\ x_2 = 2$
  $2x_1 + 3x_2 = 30$
  $\Rightarrow x = (12, 2, 3, 0, 0)$ feasible [x_1, x_2 within bounds]

(b) $x = (4, \overset{u}{6}, \overset{u}{3}, \overset{L}{0}, 4)$: x_2 at UB, x_3 at UB, x_4 at LB
x_1, x_5 basic

First check feasibility: $\quad 4 + 2(6) + 3 + 0 = 19$
$2(4) + 3(6) - 3 + 4 = 27$
also all variables within their bounds $\Rightarrow$ feasible

Next, compute reduced costs:

$$B = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \\ x_1 \quad x_5 \end{array} \qquad \pi B = c_B,\ (\pi_1\ \pi_2)\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = (5\ 0)$$

$\pi_1 + 2\pi_2 = 5$
$\pi_2 = 0$ $\quad \Rightarrow \pi = (5\ 0)$

$$\left.\begin{array}{ll} \text{(UB)} & z_2 - c_2 = (5\ 0)\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 9 = 1 > 0 \\ \text{(UB)} & z_3 - c_3 = (5\ 0)\begin{bmatrix} 1 \\ -1 \end{bmatrix} - 1 = 4 > 0 \end{array}\right\} \text{not optimal}$$

$$\text{(LB)} \quad z_4 - c_4 = (5\ 0)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 = 5 \geq 0$$

(3) $A = \begin{bmatrix} 1 & -2 & 2 & 1 & 0 & 0 \\ -2 & 1 & 6 & 0 & 1 & 0 \\ -4 & 1 & 4 & 0 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$

$c = [-7 \quad 3 \quad 9 \quad 0 \quad 0 \quad 0]$

(a) $x = (1, 9, 0)$: check feasibility

$$\begin{aligned} 1-2(9) &= -17 \le 3 & x_4 &= 20 \\ -2+9 &= 7 = 7 & x_5 &= 0 \\ -4+9 &= 5 = 5 & x_6 &= 0 \end{aligned}$$

$x = (1,9,0,20,0,0) \ge 0$ is feasible. Basic vars $\{x_1, x_2, x_4\}$

$$B = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_4 \end{array} \\ \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 0 \\ -4 & 1 & 0 \end{bmatrix} \end{array}; \quad \pi B = c_B, \quad \left.\begin{aligned} \pi_1 - 2\pi_2 - 4\pi_3 &= -7 \\ -2\pi_1 + \pi_2 + \pi_3 &= 3 \\ \pi_1 &= 0 \end{aligned}\right\} \Rightarrow \pi = (0, \tfrac{5}{2}, \tfrac{1}{2})$$

$$z_3 - c_3 = (0 \ \tfrac{5}{2} \ \tfrac{1}{2}) \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} - 9 = 17 - 9 = 8 \ge 0$$

$$z_5 - c_5 = (0 \ \tfrac{5}{2} \ \tfrac{1}{2}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = \tfrac{5}{2} \ge 0$$

$$z_6 - c_6 = (0 \ \tfrac{5}{2} \ \tfrac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \tfrac{1}{2} \ge 0$$

since all reduced costs ≥ 0, the given solution is optimal.

$z^* = -7 + 3(9) = 20$

(b) $c_2 \to 4$; this changes c_B, but not the feasibility of $x = (1, 9, 0)$.

$$\pi B = c_B, \quad \left.\begin{aligned} \pi_1 - 2\pi_2 - 4\pi_3 &= -7 \\ -2\pi_1 + \pi_2 + \pi_3 &= 4 \\ \pi_1 &= 0 \end{aligned}\right\} \Rightarrow \pi = (0, \tfrac{9}{2}, -\tfrac{1}{2})$$

$$z_3 - c_3 = (0 \ \tfrac{9}{2} \ -\tfrac{1}{2}) \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} - 9 = 16 \ge 0; \quad z_5 - c_5 = (0 \ \tfrac{9}{2} \ -\tfrac{1}{2}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 = \tfrac{9}{2} \ge 0;$$

$$z_6 - c_6 = (0 \ \tfrac{9}{2} \ -\tfrac{1}{2}) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 = -\tfrac{1}{2} < 0.$$ So not optimal; x_6 enters

Determine leaving variable using $\alpha_6 = B^{-1} a_6$, or $B\alpha_6 = a_6$:

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 0 \\ -4 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{aligned} y_1 - 2y_2 + y_3 &= 0 \\ -2y_1 + y_2 &= 0 \\ -4y_1 + y_2 &= 1 \end{aligned}\right\} \Rightarrow \alpha_6 = \begin{bmatrix} -\tfrac{1}{2} \\ -1 \\ -\tfrac{3}{2} \end{bmatrix} \le 0$$

so problem has an unbounded objective function

4 a. Add artificial variables and create Phase I problem, minimizing the sum of the artificial variables. If the optimal $z^* \neq 0$ for the Phase I problem, then the original problem is infeasible.

b. Unboundedness detected if there is some (entering) nonbasic variable x_k with $z_k - c_k < 0$ and $\alpha_k \leq 0$.

c. A BFS is degenerate if one of the basic variables has value 0.

d. Suppose x is feasible, with x_k a nonbasic variable. Optimality conditions: if x_k at LB then $z_k - c_k \geq 0$

if x_k at UB then $z_k - c_k \leq 0$