MthSc 441 - HW#2

(13) How likely is it to observe 7 matches (or at least that many)? If selected at random ($p=\frac{1}{2}$), have a binomial experiment with $n=12$:

$$P(X \geq 7) = \binom{10}{7}\left(\tfrac{1}{2}\right)^7\left(\tfrac{1}{2}\right)^3 + \binom{10}{8}\left(\tfrac{1}{2}\right)^8\left(\tfrac{1}{2}\right)^2 + \binom{10}{9}\left(\tfrac{1}{2}\right)^9\left(\tfrac{1}{2}\right)$$
$$+ \binom{10}{10}\left(\tfrac{1}{2}\right)^{10} = [120+45+10+1]\left(\tfrac{1}{2}\right)^{10}$$
$$= \frac{11}{64} \cong 0.172$$

To compare, we see how likely it is to get a result <u>at least as extreme</u> as that observed, purely by random: namely, about 17%.

(25) To require 7 games, the first six games must be evenly split (3 wins by A, 3 wins by B).

The probability of this occurring is $\binom{6}{3} p^3(1-p)^3 = 20\,p^3(1-p)^3$

To maximize $f(p) = 20\,[p(1-p)]^3$, see where derivative is 0:

$$f'(p) = 60\,[p(1-p)]^2[1-2p] = 0 \implies p = 0, 1, \tfrac{1}{2}$$

when $p = 0, 1 \implies f(p) = 0 = \min$

$* \quad p = 1/2 \implies f(p) = \frac{20}{64} = \max$

(33) a. $$1 = \int_{-1}^{1} c(1-x^2)\,dx = c\left[x - \frac{x^3}{3}\right]\Big|_{-1}^{1} = c\left(\frac{4}{3}\right) \implies \boxed{c = \frac{3}{4}}$$

b. $$F(u) = \int_{-1}^{u} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]\Big|_{-1}^{u} = \frac{1}{4}\left[3u - u^3 + 2\right]$$

for $-1 \leq u \leq 1$

NOTE: when $u=-1 \implies F(-1) = \frac{1}{4}[-3+1+2] = 0$

$u = 1 \implies F(1) = \frac{1}{4}[3-1+2] = 1$

$F(u) = 0$ for $u \leq -1$; $F(u) = 1$ for $u \geq 1$

(60) $\phi(t) = E[e^{tX}] = \int_0^1 e^{tx}\cdot 1\,dx = \frac{1}{t}e^{tx}\Big|_0^1 = \frac{1}{t}(e^t - 1)$

$\phi'(t) = \frac{(t-1)e^t + 1}{t^2}$; undeterminate $\frac{0}{0}$ for $t=0$

$\lim_{t\to 0} \frac{(t-1)e^t+1}{t^2} = \lim_{t\to 0} \frac{(t-1)e^t + e^t}{2t} = \lim_{t\to 0} \frac{te^t}{2t} = \lim_{t\to 0} \frac{e^t}{2} = \boxed{\frac{1}{2}}$

So $E[X] = \frac{1}{2}$

$\phi''(t) = \frac{(t^2-2t+2)e^t - 2}{t^3}$, also undeterminate $\frac{0}{0}$ for $t=0$

$\lim_{t\to 0} \frac{(t^2-2t+2)e^t - 2}{t^3} = \lim_{t\to 0} \frac{(t^2-2t+2)e^t + (2t-2)e^t}{3t^2}$

$= \lim_{t\to 0} \frac{t^2e^t}{3t^2} = \lim_{t\to 0} \frac{e^t}{3} = \frac{1}{3}$; so $E[X^2] = \frac{1}{3}$

$Var(X) = E[X^2] - \{E[X]\}^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{12}}$

(3) Since $\sum_x p(x,1) = \frac{5}{9}$, the conditional probabilities $P\{X|Y=1\}$ are $\frac{1}{5}, \frac{3}{5}, \frac{1}{5}$ $\Rightarrow$ $E[X|Y=1] = 1\cdot\frac{1}{5} + 2\cdot\frac{3}{5} + 3\cdot\frac{1}{5} = \boxed{2}$

Similarly $\sum_x p(x,2) = \frac{1}{6}$; $P\{X|Y=2\} = [\frac{2}{3}, 0, \frac{1}{3}]$,
$E[X|Y=2] = 1\cdot\frac{2}{3} + 2\cdot 0 + 3\cdot\frac{1}{3} = \boxed{\frac{5}{3}}$

Finally $\sum_x p(x,3) = \frac{5}{18}$, $P\{X|Y=3\} = [0, \frac{3}{5}, \frac{2}{5}]$
$E[X|Y=3] = 1\cdot 0 + 2\cdot\frac{3}{5} + 3\cdot\frac{2}{5} = \boxed{\frac{12}{5}}$

(12) $f(x,y) = \dfrac{e^{-x/y}\, e^{-y}}{y}, \quad 0 < x < \infty, \ 0 < y < \infty$

$$f_Y(y) = \int_0^\infty \frac{e^{-x/y} e^{-y}}{y}\, dx = \frac{e^{-y}}{y} \int_0^\infty e^{-x/y}\, dx = -e^{-y} e^{-x/y} \Big|_0^\infty = e^{-y}$$

So conditional density $f_{X|Y}(x|y) = \dfrac{e^{-x/y} e^{-y}}{y \cdot e^{-y}} = \dfrac{1}{y} e^{-x/y}$

which is Expon$(\lambda = \frac{1}{y})$. Thus the mean $= \frac{1}{\lambda} = \boxed{y}$. [or you can use integration by parts!]

(15) $f(x,y) = \dfrac{e^{-y}}{y}, \quad 0 < x < y, \ 0 < y < \infty$

$f_Y(y) = \displaystyle\int_0^y \frac{e^{-y}}{y}\, dx = \frac{e^{-y}}{y} \int_0^y 1 \cdot dx = e^{-y}$, so the conditional

density $f_{X|Y}(x|y) = \dfrac{e^{-y}}{y \cdot e^{-y}} = \dfrac{1}{y}$

$$E[X^2 | Y = y] = \int_0^y x^2 \cdot \frac{1}{y}\, dx = \frac{1}{y} \int_0^y x^2\, dx = \boxed{\frac{y^2}{3}}$$

(41) X = amt. of time in maze (mins)

$$E[X] = \tfrac{1}{2} \cdot \{3 + E[X]\} + \tfrac{1}{6} \cdot 2 + \tfrac{1}{3} \cdot \{5 + E[X]\} = \tfrac{7}{2} + \tfrac{5}{6} E[X]$$

$$\Rightarrow \tfrac{1}{6} E[X] = \tfrac{7}{2} \Rightarrow E[X] = \boxed{21 \text{ mins}}$$

$$E[X^2] = \tfrac{1}{2} \cdot E[3+X]^2 + \tfrac{1}{6} \cdot 2^2 + \tfrac{1}{3} \cdot E[5+X]^2$$

$$= \tfrac{1}{2} \cdot E[9 + 6X + X^2] + \tfrac{2}{3} + \tfrac{1}{3} \cdot E[25 + 10X + X^2]$$

$$= \tfrac{1}{2} \cdot \{9 + 6E[X] + E[X^2]\} + \tfrac{2}{3} + \tfrac{1}{3} \cdot \{25 + 10E[X] + E[X^2]\}$$

since $E[X] = 21 \Rightarrow$

$$= \tfrac{1}{2}\{9 + 6 \cdot 21 + E[X^2]\} + \tfrac{2}{3} + \tfrac{1}{3}\{25 + 10 \cdot 21 + E[X^2]\}$$

$$= \tfrac{293}{2} + \tfrac{1}{6} E[X^2]$$

$$\Rightarrow \tfrac{1}{6} E[X^2] = \tfrac{293}{2} \Rightarrow E[X^2] = 879$$

$$\text{var}(X) = E[X^2] - \{E[X]\}^2 = 879 - (21)^2 = \boxed{438 \text{ min}^2}$$