MthSc 441 - HW #3

(8) This is a Markov Chain with 2 states:
0 = Coin 1 is being flipped
1 = Coin 2 is being flipped

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0.67 & 0.33 \\ 0.66 & 0.34 \end{bmatrix}, \quad P^3 = \begin{bmatrix} 0.667 & 0.333 \\ 0.666 & 0.334 \end{bmatrix}$$

Using $\alpha = [0.5, 0.5] \Rightarrow$ prob of being in State 0 $= \alpha P^3 = (0.5)(0.667)+(0.5)(0.666)$
$= \boxed{0.6665}$

(20) Since the chain is irreducible and aperiodic, limiting probabilities exist and are unique. Show that $\pi = \pi P$, $\Sigma \pi_j = 1$ is solved by $\pi = [\frac{1}{M+1}, \dots, \frac{1}{M+1}]$ and we're done.

P is doubly stochastic; its column sums are 1: that is,

$[1, 1, \dots, 1] \, P = [1, 1, \dots, 1]$

As a result $\quad [\frac{1}{M+1}, \dots, \frac{1}{M+1}] \, P = [\frac{1}{M+1}, \dots, \frac{1}{M+1}]$ (*)

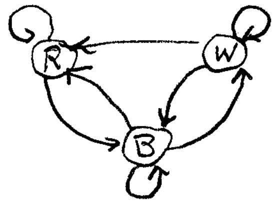
Also $\frac{1}{M+1} + \dots + \frac{1}{M+1} = 1$, so $\pi_j = \frac{1}{M+1}$ for $j = 0, 1, \dots, M$

NOTE: $\sum_0^M \pi_j = 1$ ⟶

(*) $$\boxed{\sum_{i=0}^{M} \pi_i P_{ij}} = \sum_{i=0}^{M} \left(\frac{1}{M+1}\right) P_{ij} = \frac{1}{M+1} \sum_{i=0}^{M} P_{ij} = \frac{1}{M+1}(1) = \boxed{\pi_j}$$

(24) State = $\{R, W, B\}$ ← color of ball selected

$$P = \begin{array}{c c} & \begin{array}{ccc} R & W & B \end{array} \\ \begin{array}{c} R \\ W \\ B \end{array} & \begin{bmatrix} 1/5 & 0 & 4/5 \\ 2/7 & 3/7 & 2/7 \\ 1/3 & 4/9 & 2/9 \end{bmatrix} \end{array}$$



MC is irreducible (and aperiodic), so stationary probs. exist & are unique.

Solve $\pi = \pi P$, $\sum \pi_j = 1$

$$\pi_R = \tfrac{1}{5}\pi_R + \tfrac{2}{7}\pi_W + \tfrac{1}{3}\pi_B$$

$$\pi_W = \tfrac{3}{7}\pi_W + \tfrac{4}{9}\pi_B \Rightarrow \tfrac{4}{7}\pi_W = \tfrac{4}{9}\pi_B \Rightarrow \boxed{\pi_W = \tfrac{7}{9}\pi_B}$$

$$\pi_B = \tfrac{4}{5}\pi_R + \tfrac{2}{7}\pi_W + \tfrac{2}{9}\pi_B$$

$$\tfrac{7}{9}\pi_B = \tfrac{4}{5}\pi_R + \tfrac{2}{7}\pi_W \Rightarrow \pi_W = \tfrac{4}{5}\pi_R + \tfrac{2}{7}\pi_W \Rightarrow \tfrac{5}{7}\pi_W = \tfrac{4}{5}\pi_R$$

$$\Rightarrow \pi_R = \tfrac{25}{28}\pi_W = \tfrac{25}{28}\cdot\tfrac{7}{9}\pi_B = \boxed{\tfrac{25}{36}\pi_B}$$

$$1 = \pi_R + \pi_W + \pi_B = \left(\tfrac{25}{36} + \tfrac{7}{9} + 1\right)\pi_B = \tfrac{89}{36}\pi_B \Rightarrow \pi_B = \tfrac{36}{89}$$

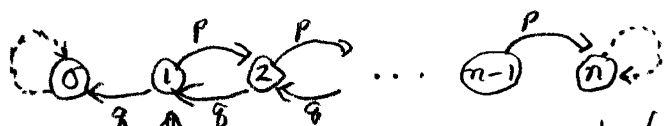
$$\Rightarrow \pi_R = \tfrac{25}{36}\cdot\tfrac{36}{89} = \tfrac{25}{89}, \quad \pi_W = \tfrac{7}{9}\cdot\tfrac{36}{89} = \tfrac{28}{89}$$

So <u>long run proportions</u> are: $\boxed{\tfrac{25}{89}\text{ (red)},\ \tfrac{28}{89}\text{ (white)},\ \tfrac{36}{89}\text{ (blue)}}$

(57) Let E be the event that, starting at 0, all states $1, 2, \ldots, n$ are visited before state 0. Condition on the first step:

$$P(E) = p \cdot P(E \mid \text{clockwise}) + q \cdot P(E \mid \text{counterclockwise}).$$

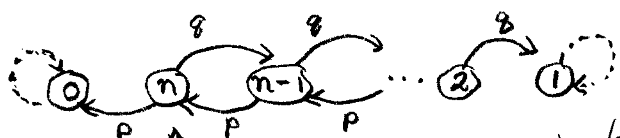
<u>clockwise</u>:



start at state 1; $y_1 = \dfrac{1 - \left(\frac{q}{p}\right)}{1 - \left(\frac{q}{p}\right)^n}$ or $y_1 = \dfrac{1}{n}$

$p \neq q \rightarrow$ $\qquad p = q \nearrow$

<u>counterclockwise</u>:



start at state n; $z_1 = \dfrac{1 - \left(\frac{p}{q}\right)}{1 - \left(\frac{p}{q}\right)^n}$ or $z_1 = \dfrac{1}{n}$

$p \neq q \rightarrow$ $\qquad p = q \nearrow$

<u>$p \neq q$</u>: $P(E) = \boxed{p \cdot \dfrac{1 - \frac{q}{p}}{1 - \left(\frac{q}{p}\right)^n} + q \cdot \dfrac{1 - \frac{p}{q}}{1 - \left(\frac{p}{q}\right)^n}}$

<u>$p = q$</u>: $\frac{1}{2}\left(\frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n}\right) = \boxed{\frac{1}{n}}$

(66) a. $P_0 = \frac{1}{4}$, $P_1 = 0$, $P_2 = \frac{3}{4}$.

$\mu = \frac{1}{4}(0) + \frac{3}{4}(2) = \frac{3}{2} > 1$; so solve $x = \frac{1}{4} + \frac{3}{4}x^2$

$4x = 1 + 3x^2$; $\quad 0 = 3x^2 - 4x + 1 = (3x-1)(x-1)$

smallest positive root is $\boxed{\pi_0 = \frac{1}{3}}$

b. $P_0 = \frac{1}{4}$, $P_1 = \frac{1}{2}$, $P_2 = \frac{1}{4}$

$\mu = \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(2) = 1 \quad \Rightarrow \boxed{\pi_0 = 1}$

c. $P_0 = \frac{1}{6}$, $P_1 = \frac{1}{2}$, $P_2 = 0$, $P_3 = \frac{1}{3}$

$\mu = \frac{1}{6}(0) + \frac{1}{2}(1) + \frac{1}{3}(3) = \frac{3}{2} > 1$; so solve equation

$x = \frac{1}{6} + \frac{1}{2}x + \frac{1}{3}x^3$; $\quad 0 = 2x^3 - 3x + 1$

Recalling that $x = 1$ is always a root gives

$0 = 2x^3 - 3x + 1 = (x-1)(2x^2 + 2x - 1)$

roots are 1, $\quad \frac{-2 \pm \sqrt{4 - 4(-2)}}{2 \cdot 2} = \frac{-1 \pm \sqrt{3}}{2}$

Smallest positive root is $\pi_0 = \frac{-1+\sqrt{3}}{2} \approx \boxed{0.366}$