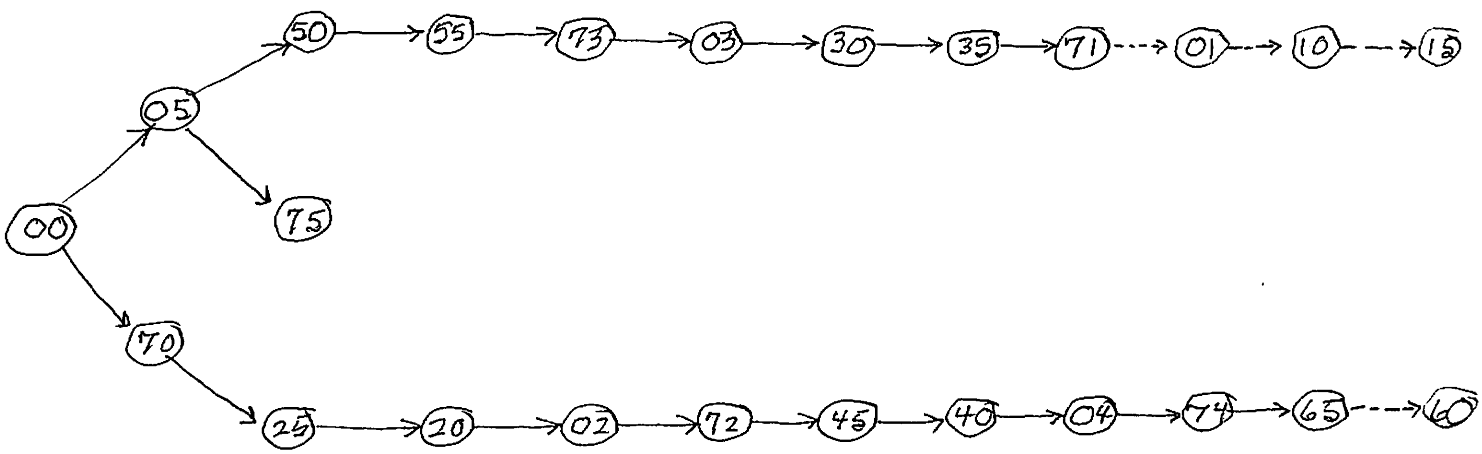
MthSc 814 – HW#1

(1) Represent each state as (i j): i gals in 7 gallon jug, j in 5 gallon jug.



(a) Shortest sequence to measure out 1 gallon (8 pourings)

00 → 05 → 50 → 55 → 73 → 03 → 30 → 35 → 71

(b) Shortest sequence to measure out 6 gallons (10 pourings)

00 → 70 → 25 → 20 → 02 → 72 → 45 → 40 → 04 → 74 → 65

(c) It's possible to measure out all integer amounts 1, 2, …, 7:

gals	min # steps
1	8
2	2
3	4
4	6
5	1
6	10
7	1

② Claim: $f(n) = \Theta(n)$

a. $f(n) = \dfrac{5n^2 - n\log n}{7n - 3\sqrt{n}} \le \dfrac{5n^2}{7n - 3\sqrt{n}} \overset{n \geq 9}{\le} \dfrac{5n^2}{7n - \sqrt{n}\sqrt{n}} = \dfrac{5n^2}{6n} = \dfrac{5}{6}n$

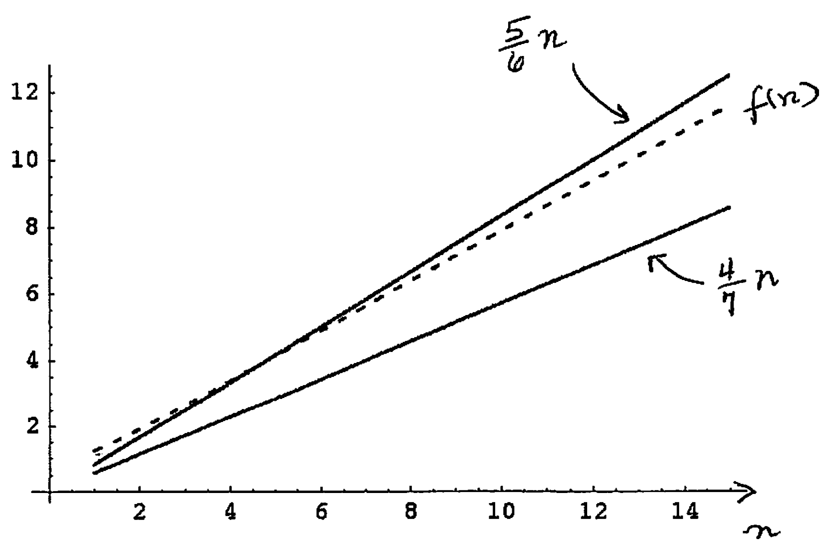
since $f(n) \le \underbrace{\tfrac{5}{6}}_{c} n$, $n \geq \underbrace{9}_{n_0}$ we have $f(n) = O(n)$

b. $f(n) = \dfrac{5n^2 - n\log n}{7n - 3\sqrt{n}} \geq \dfrac{5n^2 - n\log n}{7n} \overset{\log n \le n,\ n \geq 1}{\geq} \dfrac{5n^2 - n \cdot n}{7n} = \dfrac{5n^2 - n^2}{7n} = \dfrac{4}{7}n$

since $f(n) \geq \frac{4}{7}n$, $n \geq 1$ we have $f(n) = \Omega(n)$

Then $f(n) = \Theta(n)$: since $f(n) = O(n)$ and $f(n) = \Omega(n)$

We can verify our claim graphically:

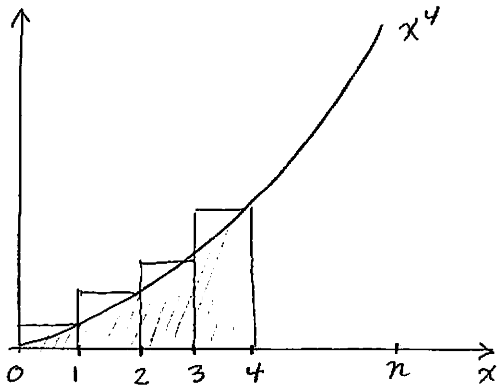


③ $f(n) = 1^4 + 2^4 + \dots + n^4$

First show $f(n) = O(n^5)$: $1^4 + 2^4 + \dots + n^4 \le n^4 + n^4 + \dots + n^4 = n \cdot n^4 = 1 n^5$ for $n \geq 1$

Next show $f(n) = \Omega(n^5)$; this can be done in several ways.

(i)



$1^4 + 2^4 + \dots + n^4 \geq \int_0^n x^4 dx = \frac{n^5}{5}$ for $n \geq 1$

(ii) Get a lower bound by adding up the "top half" of the series.

If n is even: $f(n) \geq (\frac{n}{2}+1)^4 + (\frac{n}{2}+2)^4 + \dots + n^4 \geq (\frac{n}{2}+1)^4 + \dots + (\frac{n}{2}+1)^4$
$= \frac{n}{2}(\frac{n}{2}+1)^4 \geq \frac{n}{2}(\frac{n}{2})^4 = \frac{1}{32} n^5$, for $n \geq 2$

If n is odd: $f(n) \geq (\frac{n+1}{2})^4 + (\frac{n+3}{2})^4 + \dots + n^4 \geq (\frac{n+1}{2})^4 + \dots + (\frac{n+1}{2})^4$
$= (\frac{n+1}{2})(\frac{n+1}{2})^4 \geq \frac{n}{2}(\frac{n}{2})^4 = \frac{1}{32} n^5$, for $n \geq 1$

(iii) Use induction to prove: $1^4 + 2^4 + \dots + n^4 \geq \frac{1}{5} n^5$, for $n \geq 1$

Base case ($n = 1$): $1^4 \geq \frac{1}{5} 1^5$, $1 \geq \frac{1}{5}$ ✓

Assume true for $n = k$: $1^4 + 2^4 + \dots + k^4 \geq \frac{1}{5} k^5$

Need to show: $1^4 + 2^4 + \dots + (k+1)^4 \geq \frac{1}{5}(k+1)^5$

$$\begin{aligned}
1^4 + 2^4 + \dots + k^4 + (k+1)^4 &\geq \tfrac{1}{5}k^5 + (k+1)^4 \quad \text{by inductive hyp.} \\
&= \tfrac{1}{5}k^5 + k^4 + 4k^3 + 6k^2 + 4k + 1 \\
&\geq \tfrac{1}{5}k^5 + k^4 + 2k^3 + 2k^2 + k + \tfrac{1}{5} \\
&= \tfrac{1}{5}(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) = \tfrac{1}{5}(k+1)^5
\end{aligned}$$

Thus $f(n) = \Omega(n^5)$ and so $f(n) = \theta(n^5)$.